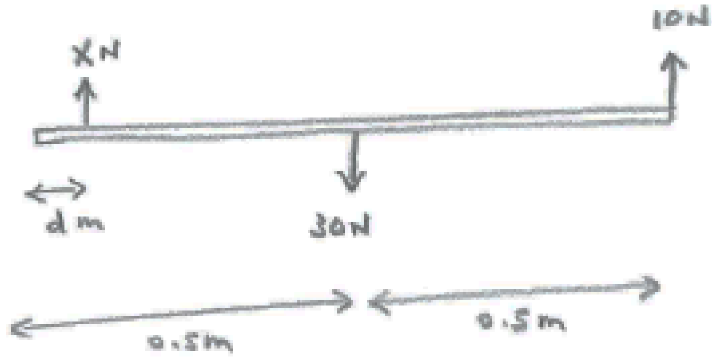
Forces - Exercises (Solutions) (30 pages; 6/4/20)

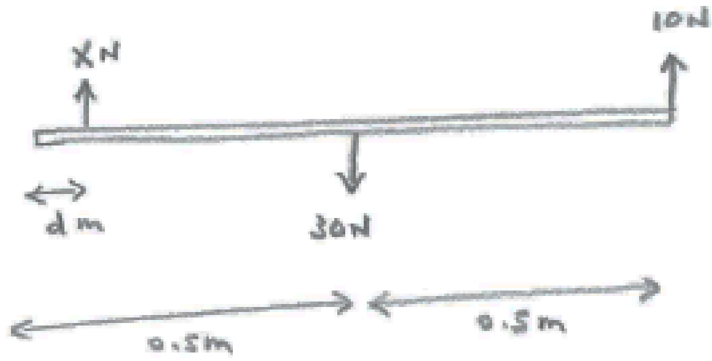
(1**) Vertical forces of X, 30 and 10 N are applied to a light rod of length 1 m, as shown in the diagram. The force of X N is applied at a distance of d m from the left-hand end, and the force of 30N is applied at the mid-point of the rod.



(a) What values must X and d have in order for the rod to be in equilbrium?

(b) The force of X N is removed, and the forces of 30N and 10N are to be replaced with a single force having the same effect as these two forces. What is the size and line of action of this single force?

Solution



(a) Vertical equilbrium $\Rightarrow X + 10 = 30 \Rightarrow X = 20$

Taking moments about the right-hand end (for example):

$30(0.5) - 20(1 - d) = 0 \Rightarrow -5 + 20d = 0 \Rightarrow d = 0.25$

[Whenever the forces are balanced, the total moment will be the same about any point; eg taking moments about the mid-point instead:

$10(0.5) - 20(0.5 - d) = 0 \Rightarrow -5 + 20d = 0$, as before

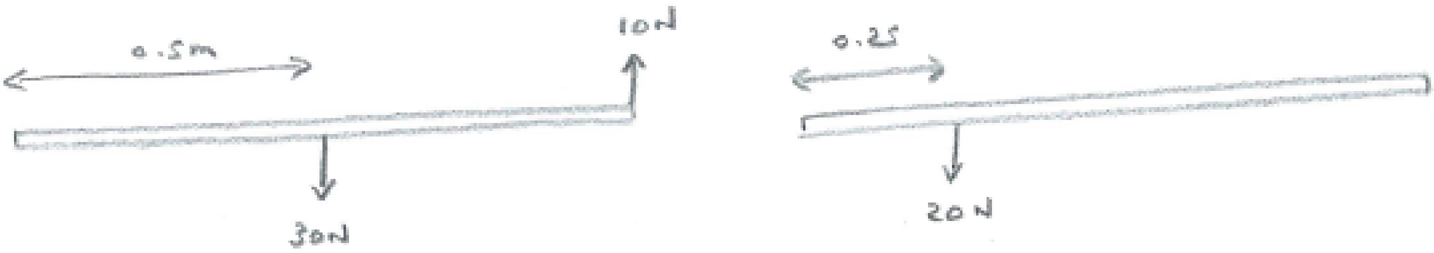
or, about the point where X is applied:

$10(1 - d) - 30(0.5 - d) = 0 \Rightarrow -5 + 20d = 0$]

(b) From (a), X counteracts the effect of the other two forces to give equilibrium.

Now a force of 20 N acting at the same position as X, but in the opposite direction, will also be counteracted by X. Hence it follows that this force is equivalent to the forces of 30 N and 10 N.

Thus the two systems shown below are equivalent.



Alternative method:

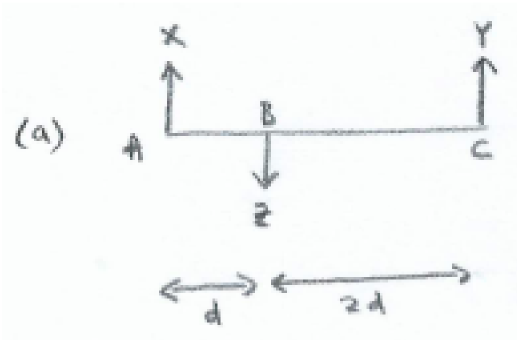
The single equivalent force must be of magnitude 20 N, acting downwards (this being the net effect of the two forces). Suppose that it acts at a distance d from the left-hand end.

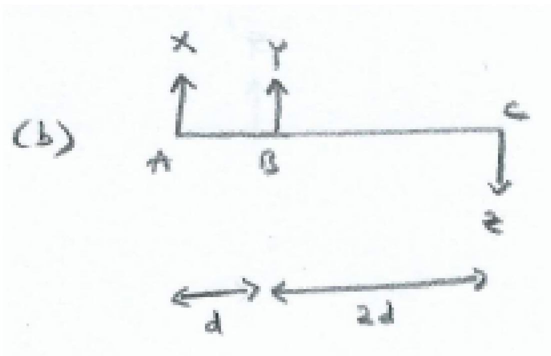
Then we require the moment of this force about the left-hand end (say) to equal the net moment of the two forces.

So $-20d = 10(1) - 30(0.5) = -5 \Rightarrow d = 0.25$

(Note: the single force could not act beyond the left-hand end, as that would give rise to a positive moment, which could not be equated to -5)

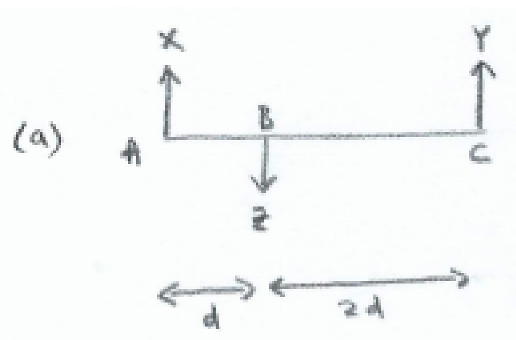
(2**) (i) Which of the following systems of forces could be in equilbrium? (with X, Y and Z > 0)

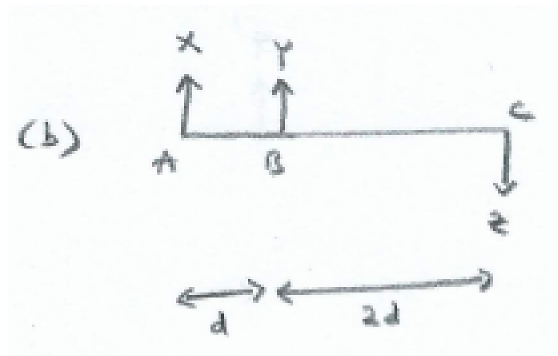




(ii) Assuming that $X + Y = Z$, show that the total moments about A, B and C are equal, in both of the cases in (i).

Solution





(i) (a) Vertical equilibrium requires that $X + Y = Z$.

For rotational equilibrium, taking moments about B, $2dY - dX = 0$, so that $X = 2Y$.

Thus there is equilibrium provided that $Y = \frac{X}{2}$ and $Z = \frac{3X}{2}$.

[Note: As about to be shown in (ii), we can take moments about any point, provided that $X + Y = Z$]

(b) If we take moments about B, we obtain $-dX - 2dZ$, which cannot equal zero. Thus the system cannot be in equilibrium.

[With 3 forces, the directions of the forces must alternate for equilibrium to be possible.]

(ii) (a) $M(A): 3dY - dZ = 3dY - d(X + Y) = d(2Y - X)$

$M(B): 2dY - dX = d(2Y - X)$

$M(C): 2dZ - 3dX = 2d(X + Y) - 3dX = d(2Y - X)$

(b) $M(A): dY - 3dZ = dY - 3d(X + Y) = -d(3X + 2Y)$

$M(B): -dX - 2dZ = -dX - 2d(X + Y) = -d(3X + 2Y)$

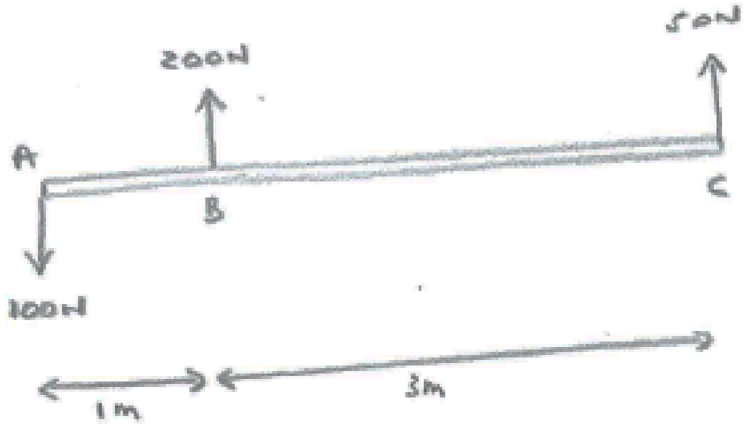
$M(C): -2dY - 3dX = -d(3X + 2Y)$

[Thus the total moment will be the same about any point, provided that the forces balance; regardless of whether there is rotational equilibrium.]

(3**) Forces are applied to a light rod, as shown in the diagram.

(a) Find the magnitude and line of action of the additional force that would be needed in order for the rod to be in equilibrium.

(b) Find the magnitude and line of action of the single force that has the same effect as the forces in the diagram.

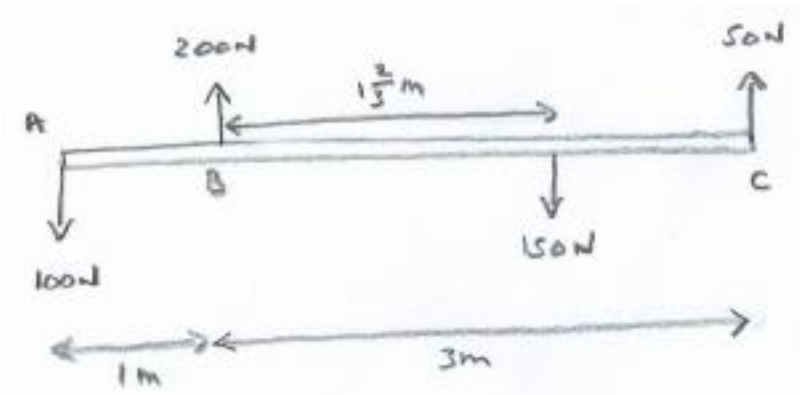


Solution

(a) In order for there to be vertical equilibrium, the additional force must be of magnitude 150N and act downwards. Suppose that its line of action is at a distance d from A.

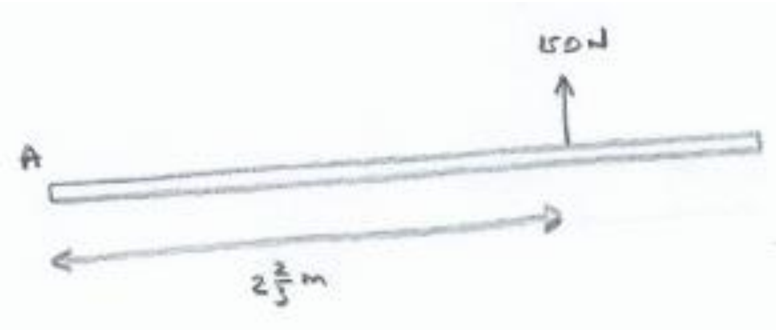
In order for there to be rotational equilibrium, the net moment about A (say) must be zero; ie

$200(1) + 50(4) - 150d = 0 \Rightarrow d = \frac{400}{150} = \frac{8}{3}\,\mathrm{m}$



(b) **Method 1**

From (a), the force of 150N counteracts the other forces to create equilibrium. This force of 150N also counteracts an equal and opposite force acting at the same point. Hence, this equal and opposite force must have the same effect as the forces in the diagram; ie the required force is as shown below:



Method 2

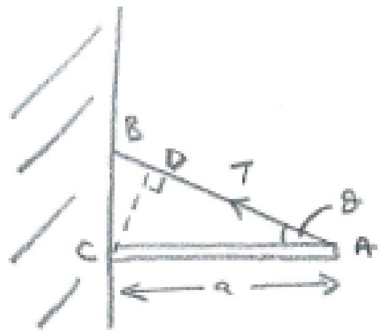
The required force must have the same net value and direction as the forces in the diagram. It therefore has value 150N and acts upwards.

This force will also have the same moment about A (or any other point) as the forces in the diagram.

Suppose that its line of action is at a distance d from A.

Then $150d = 200(1) + 50(4) \Rightarrow d = \frac{400}{150} = \frac{8}{3}$ m

(4**)



Show that the moment of T about C is the same:

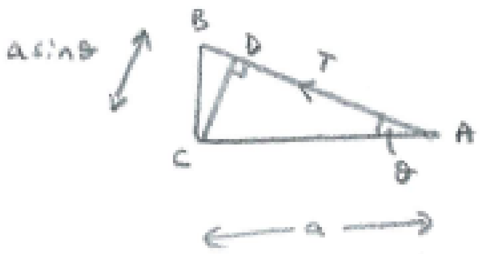
(i) if T is multiplied by CD

(ii) T is resolved into horizontal & vertical components at A

(iii) T is resolved into horizontal & vertical components at B

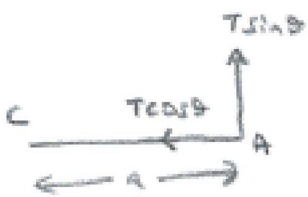
Solution

(i)



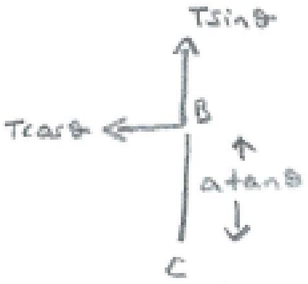
moment $= T \times CD = Tasin\theta$

(ii)



moment $= (Tcos\theta)(0) + (Tsin\theta)a = Tasin\theta$

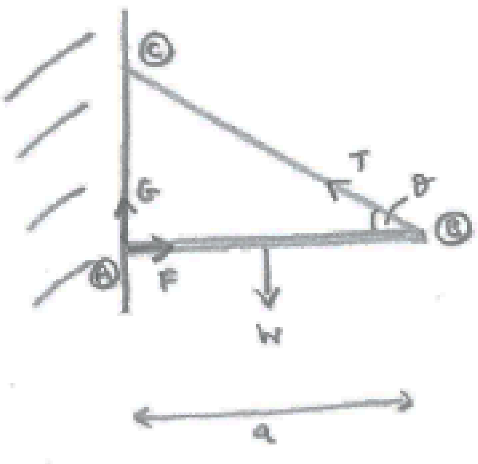
(iii)



Referring to the original diagram, $CB = atan\theta$, so that

moment = $(Tcos\theta)(atan\theta) + (Tsin\theta)(0) = Tasin\theta$

(5) Alternative Moments Methods**



A rod AB is attached to a wall at A, and held in a horizontal position by a rope BC.

Show that, as an alternative to resolving forces horizontally and vertically, and taking moments about A, it is also possible to:

(a) resolve forces horizontally and take moments about A & B,

or (b) take moments about A, B & C;

but that it is not possible to do the following:

(c) resolve forces vertically and take moments about A & B,

or (d) take moments about A, B & the midpoint of AB

Solution

Resolving forces horizontally and vertically,

$F = Tcos\theta$ (1) & $W = G + Tsin\theta$ (2)

Taking moments about A gives $(Tsin\theta)a - W\left(\frac{a}{2}\right) = 0$ (3)

Then (3) $\Rightarrow T = \frac{W}{2sin\theta}$

and hence (1) $\Rightarrow F = \frac{Wcot\theta}{2}$

and (2) $\Rightarrow G = W - \frac{W}{2} = \frac{W}{2}$

Following method (a) instead,

resolving horizontally gives $F = Tcos\theta$ (4);

taking moments about A gives $(Tsin\theta)a - W\left(\frac{a}{2}\right) = 0$ (5),

and taking moments about B gives $-Ga + W\left(\frac{a}{2}\right) = 0$ (6)

Then from (6), $G = \frac{W}{2}$;

from (5), $T = \frac{W}{2sin\theta}$,

and from (4), $F = \frac{Wcot\theta}{2}$

Following method (b) instead,

taking moments about A gives $(Tsin\theta)a - W\left(\frac{a}{2}\right) = 0 \quad (7)$;

taking moments about B gives $-Ga + W\left(\frac{a}{2}\right) = 0 \quad (8)$,

and taking moments about C gives

$F(atan\theta) - W\left(\frac{a}{2}\right) = 0 \quad (9)$

Then from (8), $G = \frac{W}{2}$;

from (7), $T = \frac{W}{2sin\theta}$,

and from (9), $F = \frac{Wcot\theta}{2}$

However, following method (c):

resolving vertically gives $W = G + Tsin\theta \ \ (10)$;

taking moments about A gives $(Tsin\theta)a - W\left(\frac{a}{2}\right) = 0 \quad (11)$,

and taking moments about B gives $-Ga + W\left(\frac{a}{2}\right) = 0 \quad (12)$,

but we have no equation involving F

Also, following method (d):

taking moments about A gives $(Tsin\theta)a - W\left(\frac{a}{2}\right) = 0 \quad (13)$;

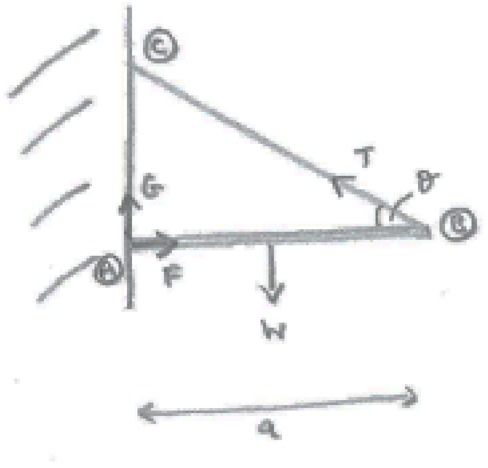
taking moments about B gives $-Ga + W\left(\frac{a}{2}\right) = 0 \quad (14)$,

and taking moments about the midpoint of AB gives

$-G\left(\frac{a}{2}\right) + (Tsin\theta)\left(\frac{a}{2}\right) = 0$,

and once again we have no equation involving F

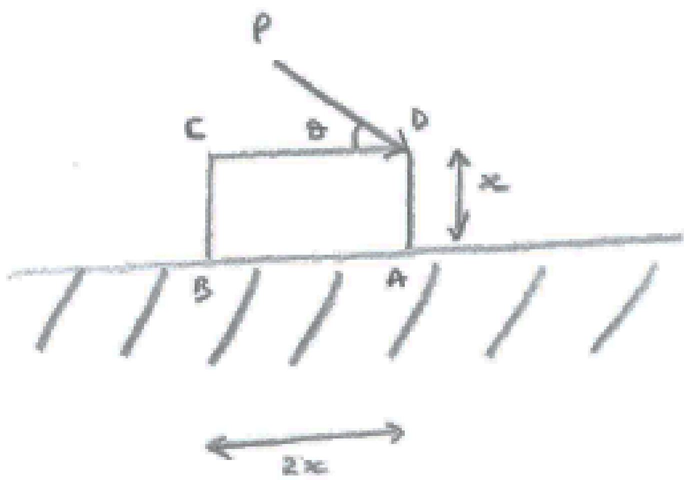
(6*) Referring to the diagram below, about which point should moments be taken, in order to find F in terms of W?



Solution

As the lines of action of G & T (the two unwanted forces) pass through C, this is the required point.

(7***) A uniform block of mass m rests on a table, and a force P is applied at D, as shown in the diagram. The block has length $2x$ and height x. The coefficient of friction between the block and the table is μ.



(i) If the block is on the point of sliding, find an expression for P.

(ii) If instead the block is on the point of toppling, find an expression for P.

(iii) If the block is to topple before it slides, find a condition on μ.

Solution

(i) The normal reaction, $R = mg + Psin\theta$

The frictional force $= \mu(mg + Psin\theta)$

Hence, at the point of sliding, $\mu(mg + Psin\theta) = Pcos\theta$,

so that $P(cos\theta - \mu sin\theta) = \mu mg$

and $P = \frac{\mu mg}{cos\theta - \mu sin\theta}$

(ii) If the block is on the point of toppling, it will be about A, and the only reaction on the block will be at A. [This will be a combination of a normal reaction and friction.]

As the block is uniform, its weight will act at a distance x from AD, and so, taking moments about A,

$(mg)x = (Pcos\theta)x$

[the normal reaction and friction contribute nothing, as they act at A]

Hence $P = \frac{mg}{cos\theta}$

(iii) At the critical position where the block is about to both slide and topple,

$P = \frac{\mu mg}{cos\theta - \mu sin\theta} = \frac{mg}{cos\theta}$

so that $\mu cos\theta = cos\theta - \mu sin\theta$;

$\mu(cos\theta + sin\theta) = cos\theta$

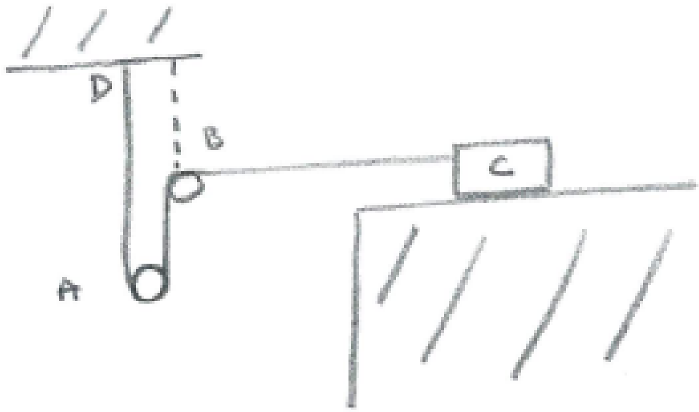
and $\mu = \frac{cos\theta}{cos\theta + sin\theta} = \frac{1}{1+tan\theta}$

So, if the block is to topple before it slides, we require

$\mu > \frac{1}{1+tan\theta}$ [ie making the frictional force greater]

[reasonableness check: if $\theta = 45°$, then $\mu > 0.5$; also, if θ is reduced to 30°, we would expect a higher value of μ to be necessary, in order for toppling to occur first (since the block is now more prone to slide than topple), and the condition gives $\mu > \frac{1}{1+\frac{1}{\sqrt{3}}} = 0.634$]

(8***)



Referring to the diagram, A is a smooth pulley of mass 2 kg, which can move up and down; B is a smooth, fixed pulley, and C is a block of mass 1kg, which is initially held at rest on a table. A light inextensible rope is fixed at D, and leads to C, via the two pulleys.

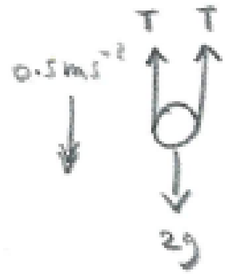
C is now released and accelerates at 2 ms^{-1}. Find the coefficient of friction, μ between C and the table.

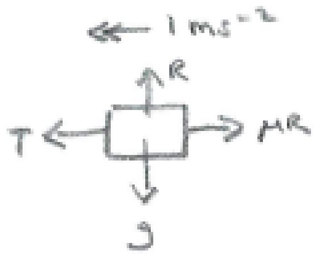
Solution

First of all, the tension (T, say) is the same throughout the rope (since the rope is light and the pulleys are smooth - see note (i) below).

Also, because A falls by half the distance that C moves, its acceleration is also half that of C - see note (ii) below).

[The fact that the rope is inextensible ensures that the rope (as a whole) and the block move the same distances.]





From the force diagram for A,

N2L ⇒ $2g - 2T = (2)(1) \Rightarrow T = g - 1 \quad [1]$

From the force diagram for B,

N2L ⇒ $T - \mu R = (1)(2)$

Also, vertical equilibrium ⇒ $R = g$,

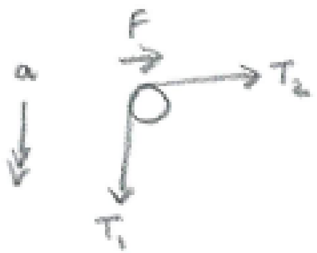
so that $\quad T - \mu g = 2 \quad [2]$

Then [1]&[2] ⇒ $\mu g = g - 3$,

so that $\mu = 1 - \frac{3}{9.8} = 0.694$ (3sf)

Notes

(i) Referring to the force diagram for the rope around B, for example:



Suppose that the rope has mass m, that the pulley exerts a frictional force F, and that the rope experiences forces T_1 and T_2 at its ends.

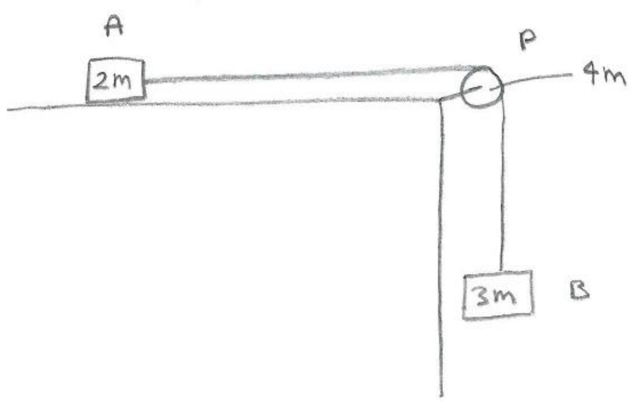
If the rope has acceleration a,

then $T_1 - T_2 - F = ma$

If $F = 0$ (ie if the pulley is smooth), and $m \approx 0$ (ie the rope is 'light', and so has negligible mass), then $T_1 - T_2 \approx 0$, and so the tensions are approximately equal.

(ii) From the suvat equation, if $u = 0$, then $s = \frac{1}{2}at^2$, so that the acceleration is proportional to the distance, for a given t.

(9***) Rotating Pulley



Initially block A is held at rest on a smooth table. The pulley P can rotate freely. The string leading from A to B, passing over P, is light and inextensible.

The pulley is a uniform disc of radius r, and the blocks can be modelled as particles.

Block A is released. The tension in the section of the string AP is T_A and in PB it is T_B.

Assuming that the string does not slip on the pulley, and that A does not reach P,

(i) Show that the angular acceleration of the pulley is $\frac{3g}{7r}$ $rads^{-2}$

(ii) Find T_A and T_B in terms of m and g.

Solution

The moment of inertia, I of P is $\frac{1}{2}(4m)r^2 = 2mr^2$ [standard result for a disc about its axis]

The total moment of the external forces on P about its axis, $C = I\ddot{\theta}$, where $\ddot{\theta}$ is the angular acceleration of P.

$C = T_B r - T_A r$

Hence $r(T_B - T_A) = 2mr^2\ddot{\theta}$ (1)

The acceleration of A and B is $r\ddot{\theta}$ [the distance fallen by $B = r\theta$ (the arc length travelled by a point on the circumference of the pulley), and this is differentiated twice]

So, for A, N2L $\Rightarrow T_A = (2m)(r\ddot{\theta})$ (2)

and for B: $(3m)g - T_B = (3m)(r\ddot{\theta})$ (3)

Substituting for T_A and T_B from (2) & (3) into (1):

$\left(3mg - 3mr\ddot{\theta}\right) - 2mr\ddot{\theta} = 2mr\ddot{\theta}$

so that $7mr\ddot{\theta} = 3mg$, and $\ddot{\theta} = \frac{3g}{7r}$ $rads^{-2}$

Alternative method

By Conservation of energy,

$$\frac{1}{2}I\left(\dot{\theta}\right)^2+\frac{1}{2}(2m)\left(r\dot{\theta}\right)^2+\frac{1}{2}(3m)\left(r\dot{\theta}\right)^2-(3m)g(r\theta)=\text{constant}$$

(taking the initial position of B as the zero of PE; as before, $r\theta$ is the distance that B has fallen when P has rotated by θ rad)

Hence $\left\{\left(\frac{1}{2}\right)2mr^2+\left(\frac{5}{2}\right)mr^2\right\}\left(\dot{\theta}\right)^2-3mgr\theta=\text{constant}$

Differentiating wrt time,

$\left(\frac{7}{2}\right)mr^2(2)\dot{\theta}\ddot{\theta}-3mgr\dot{\theta}=0,$

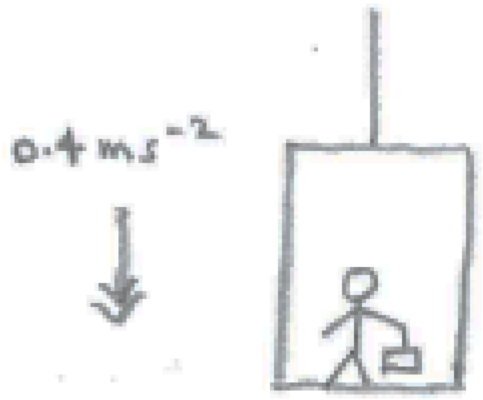
so that $7r\ddot{\theta}-3g=0,$ and $\ddot{\theta}=\frac{3g}{7r}\quad rads^{-2}$

(ii) From (2), $T_A=2mr\left(\frac{3g}{7r}\right)=\frac{6mg}{7}$

From (3), $T_B=3mg-3mr\left(\frac{3g}{7r}\right)=\frac{12mg}{7}$

(10***) Forces

A man is in a lift, which is moving downwards with an acceleration of $0.4ms^{-1}$. The lift is suspended by a cable, and the man is holding a parcel by a light string, as in the diagram. The masses of the lift, man and parcel are 300kg, 80kg and 5kg, respectively.



(i) Find :

(a) the tension in the cable

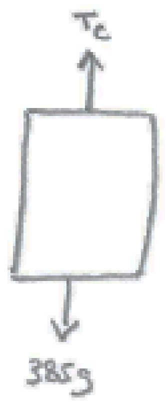
(b) the reaction between the man and the floor of the lift

(c) the tension in the string

(ii) Does the man feel heavier or lighter than he would if the lift were stationary and he were no longer carrying the parcel?

Solution

(a) Considering the lift, man and parcel as a single object:



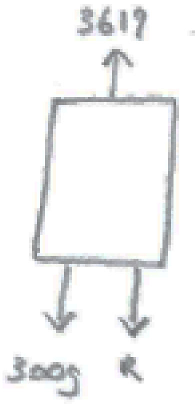
[T_C and 385g are the external forces]

$N2L \Rightarrow 385g - T_C = 385(0.4)$, where T_C is the tension in the cable

[any new symbols introduced need to be defined in an exam answer]

$\Rightarrow T_C = 385(9.8 - 0.4) = 3619\ N$

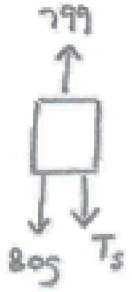
(b) Considering the forces on the lift:



$N2L \Rightarrow 300g + R - 3619 = 300(0.4)$, where R is the reaction between the man and the floor

$\Rightarrow R = 3619 + 120 - 300(9.8) = 799N$

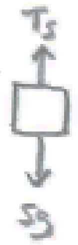
(c) Considering the forces on the man:



$N2L \Rightarrow 80g + T_S - 799 = 80(0.4)$, where T_S is the tension in the string

$\Rightarrow T_S = 799 + 32 - 80(9.8) = 47N$

[Check: Considering the forces on the parcel:



$N2L \Rightarrow 5g - T_S = 5(0.4)$

$\Rightarrow T_S = 5(9.8) - 2 = 47N$]

(ii) If the lift is stationary and the man is not carrying the parcel, the reaction between himself and the floor is just his weight [see note below]: $80(9.8) = 784N$

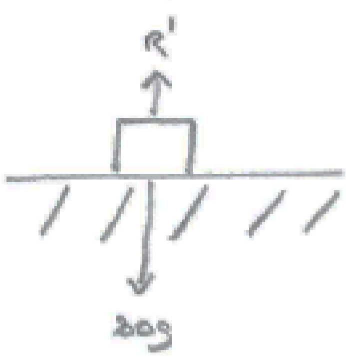
Thus he feels heavier, as $799 > 784$.

[The apparent gravity is now $9.8 - 0.4 = 9.4$, but the man's weight has effectively been increased by 5kg, giving a net apparent weight of

$85 \times 9.4 = 799N$ (this is a check on (b))]

Note: In the stationary situation (with no parcel),

$N2L \Rightarrow 80g - R' = 0 \Rightarrow R' = 80g$; ie the man's weight



(11**) Friction

A sledge with a child onboard is being pulled along on level ground, at a constant speed, by means of a rope inclined at 30° to the horizontal. The sledge and child together have a mass of $100kg$. The coefficient of friction between the sledge and the ground is $\frac{1}{10}$. Assuming that $g = 10$, find the tension in the rope.

Solution

Let T be the tension, and let R be the normal reaction of the ground on the sledge. Then, applying N2L vertically:

$R + Tsin30° = 100g$

Applying N2L horizontally, $Tcos30° = \mu R$

Hence $T\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{10}\left(1000 - \frac{T}{2}\right)$,

so that $T\left(\frac{\sqrt{3}}{2} + \frac{1}{20}\right) = 100$

and $T = 109\ N\ (3sf)$

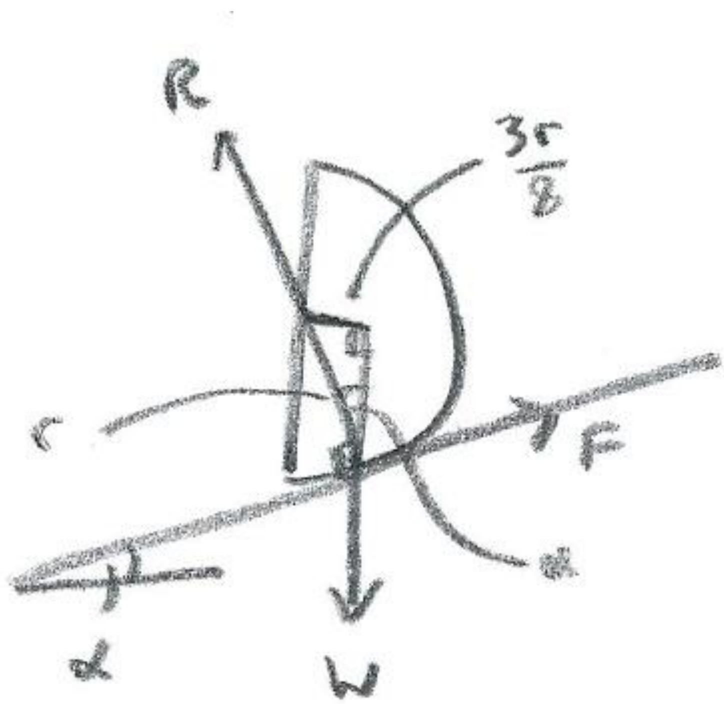
(12***) A uniform solid hemisphere rests in equilibrium on a rough slope, with its curved surface in contact with the slope, which is inclined at an angle α to the horizontal, in such a way that the plane face of the hemisphere is vertical. Find α.

Solution

In order to establish the necessary configuration of the hemisphere and slope, we note that the weight of the hemisphere must act on a line that passes through the point of contact between the hemisphere and the slope. [The three forces acting on the hemisphere (its weight, the reaction from the slope and friction) will then all meet at a single point, as is required for a body in equilibrium that is subject to three forces - otherwise a non-zero moment would exist about the point of intersection of two of the forces).

A diagram can be drawn by starting with the hemisphere, and adding in the slope. Note also that the point of contact will be on a tangent to the hemisphere, and that the perpendicular to the tangent, along which the reaction force acts, will be a radius of the hemisphere.

The radius of the hemisphere (which we are expecting to cancel out, as it isn't mentioned in the question) can be taken to be r.



The weight can be taken to act at the centre of mass of the hemisphere, which is at a distance $\frac{3r}{8}$ from the plane face.

From the diagram, $sin\alpha = \frac{\left(\frac{3r}{8}\right)}{r} = \frac{3}{8}$, and hence $\alpha = 22.0°$ (1dp).

(13***) A stepladder is made up of two sides, which have weights 80N and 8N. Both sides are of length 2m. There is a platform resting on the top, which together with a person standing on it weighs 700N. The two sides are also joined together by a horizontal light rope of length 1m, which starts at a distance of 0.6m along each side, from the base. See Fig. 1. There is no friction between the ladder and the ground, or between the platform and the ladder. Find the tension in the rope.

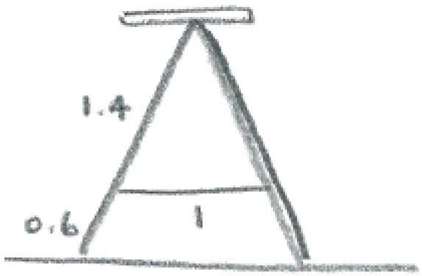


Fig. 1

Solution

[Equations can be obtained by applying N2L and/or taking moments for either individual components (eg one side of the stepladder), or the whole system. There will be a limit to the number of independent equations that can be created. But because some equations may prove to be redundant anyway (as we will see), it may not be worth attempting to ensure that all the equations are independent. Instead, we can just create any equations that look useful, and run the risk of some duplication. Not all of the equations created below are needed for the purpose of finding T, but may be of use for establishing other forces - or combinations of forces.]

Fig. 1a shows the various lengths involved.

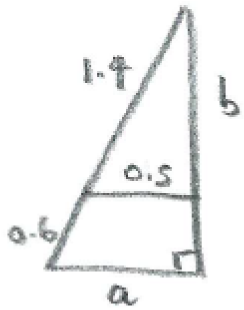


Fig. 1a

Considering the external forces on the ladder (including the rope, but excluding the platform) (Fig. 2),

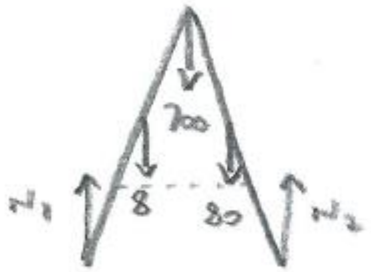


Fig. 2

$N_1 + N_2 = 788$ (1)

Also, taking moments about the base of the righthand side of the ladder :

$-N_1(2a) + (8)\left(\frac{3a}{2}\right) + 700a + 80\left(\frac{a}{2}\right) = 0$ (2),

so that $N_1 = \frac{1}{2}(12 + 700 + 40) = 376$, and hence $N_2 = 412$

Fig. 3 shows the force diagram for the lefthand side of the ladder (excluding the rope and the platform), where X & Y are the components of the reaction from the righthand side of the ladder, and R_1 is the reaction from the platform and person.

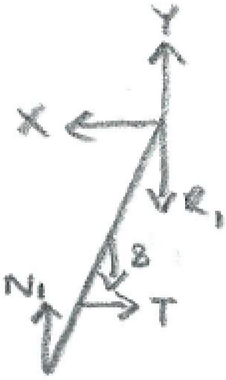


Fig.3

Horizontally this gives $T = X$ (3)

and vertically: $N_1 + Y = 8 + R_1$,

As $N_1 = 376$, $368 + Y = R_1$ (4)

By N3L, the reaction forces on the righthand side of the ladder from the lefthand side are X & Y, as shown in Fig.4. And R_2 is the reaction from the platform and person.

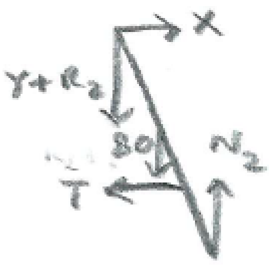


Fig. 4

Then, vertically: $N_2 = 80 + Y + R_2$

As $N_2 = 412$, $332 = Y + R_2$ (5)

(Horizontally just reproduces $T = X$)

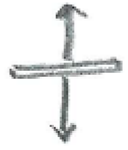


Fig.5

Fig. 5 shows the forces on the platform and person, giving

$R_1 + R_2 = 700$ (6)

[Note that this could have been obtained by adding (4) & (5).]

In order to involve the location of T, we can take moments about the top of the ladder, for say the lefthand side, to give:

$-N_1 a + Tb + (8)\left(\frac{a}{2}\right) = 0$ (6),

so that $T = \frac{a}{b}(376 - 4) = \frac{372a}{b}$

Also, from Fig. 1a, by similar triangles, $\frac{a}{0.5} = \frac{2}{1.4}$, so that $a = \frac{5}{7}$,

and $b^2 = 1.4^2 - 0.5^2$, so that $b = \frac{\sqrt{171}}{10} = \frac{3\sqrt{19}}{10}$,

and hence $T = \frac{372(5)(10)}{7(3\sqrt{19})} = 203.197$

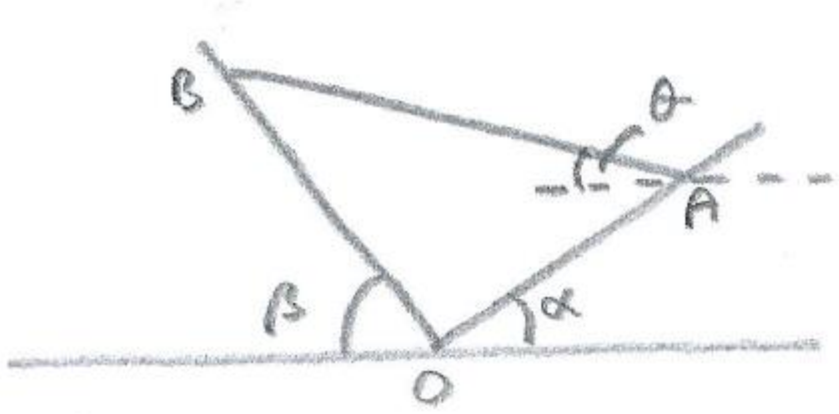
(14***) [from Wragg: "Modern Mechanics - A vectorial approach"]

A uniform rod AB lies in equilibrium between two smooth planes inclined at angles α and β to the horizontal, as shown in the diagram, where $\beta > \alpha$, such that the vertical plane containing AB is perpendicular to the line of intersection of the two planes.

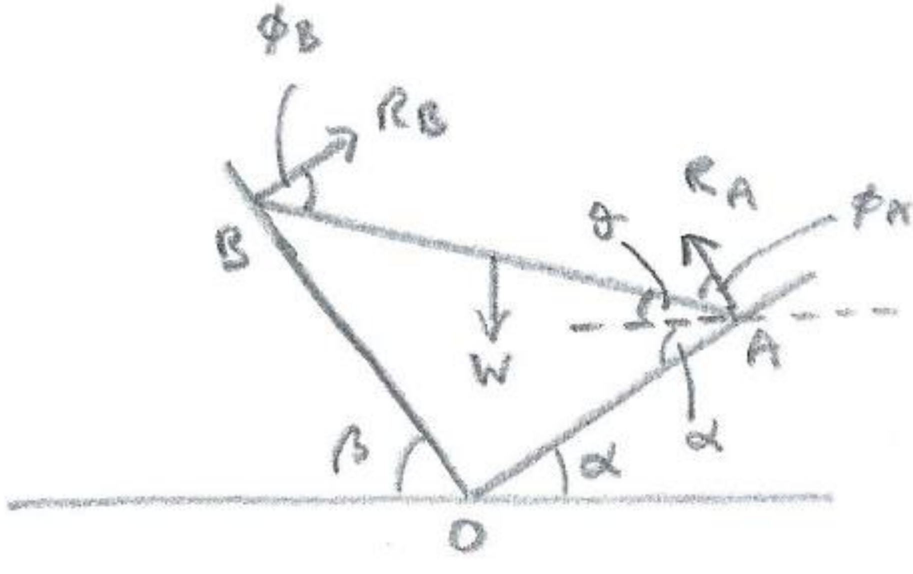
(i) Show that the ratio of the reactions at A and B is $sin\beta : sin\alpha$

(ii) If AB makes an angle θ to the horizontal, show that

$tan\theta = \frac{\sin(\beta-\alpha)}{2sin\alpha sin\beta}$



Solution



(i) Taking moments about the centre of mass of AB, which is assumed to be of length $2d$,

rotational equilibrium $\Rightarrow R_A sin\phi_A d = R_B sin\phi_B d$,

And $\phi_A + \theta + \alpha = 90°$ (the angle that R_A makes with OA)

& $\phi_B + (180° - [180° - \alpha - \beta] - [\alpha + \theta]) = 90°$

(the angle that R_B makes with OB),

and so $\phi_B - \theta + \beta = 90°$

Then $\frac{R_A}{R_B} = \frac{sin\phi_B}{sin\phi_A} = \frac{cos(90°-\phi_B)}{cos(90°-\phi_A)} = \frac{cos(\beta-\theta)}{cos(\alpha+\theta)}$ (1)

[Instead of resolving in 2 perpendicular directions, we can (if necessary) obtain 2 equations from Lami's theorem:]

As AB is in equilbrium, the triangle of forces can be applied (see diagram, where W is the weight of AB). Then, by Lami's theorem:

$\frac{R_A}{sin\gamma_A} = \frac{R_B}{sin\gamma_B}$ (2)

[A similar equation involving W could also be obtained, but this introduces a further unknown (ie W) into the equations.]

Drawing a vertical line through B gives

$\gamma_A + 90° + (90° - \beta) = 180°$, so that $\gamma_A = \beta$

Drawing a vertical line through A gives

$\gamma_B + 90° + (90° - \alpha) = 180°$, so that $\gamma_B = \alpha$

Then, from (2), $\frac{R_A}{R_B} = \frac{sin\gamma_A}{sin\gamma_B} = \frac{sin\beta}{sin\alpha}$, as required.

(ii) From (1), $\frac{cos(\beta-\theta)}{cos(\alpha+\theta)} = \frac{sin\beta}{sin\alpha}$, so that

$sin\alpha(cos\beta cos\theta + sin\beta sin\theta) = sin\beta(cos\alpha cos\theta - sin\alpha sin\theta)$

and hence (dividing by $cos\theta$),

$sin\alpha cos\beta + sin\alpha sin\beta tan\theta = sin\beta cos\alpha - sin\beta sin\alpha tan\theta$,

so that $tan\theta(2sin\alpha sin\beta) = sin\beta cos\alpha - sin\alpha cos\beta$,

and $tan\theta = \frac{\sin(\beta-\alpha)}{2sin\alpha sin\beta}$, as required.