

Transformations - Further Exercises

(10 pages; 28/12/24)

(i) What series of transformations is equivalent to a reflection in the line $x = L$?

(ii) What is the effect of a reflection in the line $x = L$ on the function $y = f(x)$?

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Solution

(i) Translation of $\begin{pmatrix} -L \\ 0 \end{pmatrix}$, followed by reflection in y -axis, followed by translation of $\begin{pmatrix} L \\ 0 \end{pmatrix}$

(ii) Translation of $\begin{pmatrix} -L \\ 0 \end{pmatrix}$: $y = f(x) \rightarrow y = f(x + L)$;

reflection in y -axis: $y = f(x + L) \rightarrow y = f(-x + L)$

translation of $\begin{pmatrix} L \\ 0 \end{pmatrix}$: $y = f(-x + L) \rightarrow y = f(-[x - L] + L)$

$= f(2L - x)$

What combination of transformations converts $y = 2^x$ to
 $y = 2^{4x-2}$?

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$$y = 2^{4x-2}?$$

Solution

$y = 2^x \rightarrow y = 2^{4x}$ is a stretch of scale factor $\frac{1}{4}$ in the x -direction

Then $y = 2^{4x} \rightarrow y = 2^{4(x-\frac{1}{2})} = 2^{4x-2}$ is a translation of $\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$

[Alternatively, $y = 2^{4x} \rightarrow y = \left(\frac{1}{4}\right) 2^{4x} = 2^{4x-2}$ is a stretch of scale factor $\frac{1}{4}$ in the y -direction.]

(i) Find a series of transformations that can be applied to $y = \frac{1}{x}$ to produce $y = \frac{3x-2}{6x-1}$.

(ii) Sketch the curve $y = \frac{3x-2}{6x-1}$.

(i) Find a series of transformations that can be applied to $y = \frac{1}{x}$ to produce $y = \frac{3x-2}{6x-1}$.

(ii) Sketch the curve $y = \frac{3x-2}{6x-1}$

Solution

$$(i) \frac{3x-2}{6x-1} = \frac{3x - \frac{1}{2} - \frac{3}{2}}{6x-1} = \frac{1}{2} - \frac{3}{12} \left(\frac{1}{x - \frac{1}{6}} \right)$$

So a possible series of transformations is:

a translation of $\begin{pmatrix} \frac{1}{6} \\ 0 \end{pmatrix}$,

followed by a stretch of scale factor $\frac{1}{4}$ in the y -direction,

followed by a reflection in the x -axis,

followed by a translation of $\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$

[Note: $\frac{1}{2} - \frac{3}{12} \left(\frac{1}{x - \frac{1}{6}} \right) = \frac{1}{2} - \frac{1}{4x - \frac{2}{3}}$, so an alternative series of transformations is:

a translation of $\begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$ [$\frac{1}{x} \rightarrow \frac{1}{x - \frac{2}{3}}$] followed by

a stretch of scale factor $\frac{1}{4}$ in the x -direction [$\frac{1}{x - \frac{2}{3}} \rightarrow \frac{1}{4x - \frac{2}{3}}$], followed

by a reflection in the x -axis, followed by a translation of $\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$.

Alternatively, $\frac{1}{4x-\frac{2}{3}}$ could be obtained instead by a stretch of scale factor $\frac{1}{4}$ in the x -direction $[\frac{1}{x} \rightarrow \frac{1}{4x}]$ (or a stretch of scale factor $\frac{1}{4}$ in the y -direction

$[\frac{1}{x} \rightarrow \frac{1}{4}(\frac{1}{x})]$), followed by a translation of $\begin{pmatrix} \frac{1}{6} \\ 0 \end{pmatrix}$ $[\frac{1}{4x} \rightarrow \frac{1}{4(x-\frac{1}{6})}]$.]

(ii) As an alternative to performing the transformations in (i):

$$\text{Step 1: } x = 0 \Rightarrow y = 2 ; y = 0 \Rightarrow x = \frac{2}{3}$$

$$\text{Step 2: vertical asymptote when } 6x - 1 = 0 \Rightarrow x = \frac{1}{6}$$

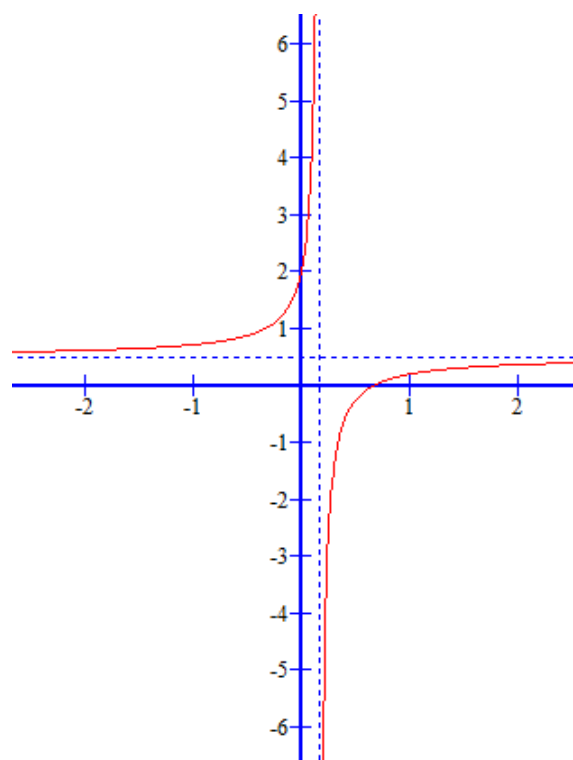
$$x = \frac{1}{6} + \delta \text{ } (\delta > 0 \text{ is small}) \Rightarrow y = \frac{3x-2}{6x-1} = \frac{-}{+}; \text{ ie } y < 0$$

$$[x = \frac{1}{6} - \delta \Rightarrow y = \frac{-}{-}; \text{ ie } y > 0]$$

$$\text{Step 3: } \lim_{x \rightarrow \infty} \frac{3x-2}{6x-1} = \lim_{x \rightarrow \infty} \frac{3-\frac{2}{x}}{6-\frac{1}{x}} = \frac{3}{6} = \frac{1}{2} \text{ (and also as } x \rightarrow -\infty)$$

$$\text{Step 4: When } x = 100, y = \frac{298}{599} < \frac{1}{2}, \text{ so that } y \rightarrow \frac{1}{2}^- \text{ as } x \rightarrow \infty$$

$$\text{and when } x = -100, y = \frac{-302}{-601} > \frac{1}{2}, \text{ so that } y \rightarrow \frac{1}{2}^+ \text{ as } x \rightarrow -\infty$$



What combination of transformations converts $y = 3^{-x}$ to $y = 3^{2x-1}$?

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Solution

$y = 3^{-x} \rightarrow y = 3^x$ is a reflection in the y -axis.

$y = 3^x \rightarrow y = 3^{x-1}$ is a translation of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$y = 3^{x-1} \rightarrow y = 3^{2x-1}$ is a stretch of scale factor $\frac{1}{2}$ in the x -direction