Inequalities - Ideas & Exercises (20 pages; 29/12/24)

Prove that, for $a, b, c > 0$, $\frac{a}{b} < \frac{a+c}{b+c} \Leftrightarrow a < b$

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Solution

$\frac{a}{b} < \frac{a+c}{b+c} \Leftrightarrow a(b + c) < b(a + c)$

$\Leftrightarrow ac < bc \Leftrightarrow a < b$

Outline possible methods for solving the inequality $\frac{1}{x} < x$

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Solution

Method 1: Multiply both sides by x^2, and consider critical points

Method 2: Treat the cases $x < 0$ and $x > 0$ separately

Method 3: Rearrange as $\frac{1}{x} - x < 0$, and consider critical points

Method 4: Sketch $y = \frac{1}{x}$ and $y = x$, and consider points of intersection

Solve $\frac{x-2}{x-1} < 3$

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Solution

Method 1 ('Case by case')

Case 1: $x - 1 > 0$

$\Rightarrow x - 2 < 3x - 3$

$\Rightarrow 1 < 2x \quad \Rightarrow x > \frac{1}{2}$

$x - 1 > 0 \ \& \ x > \frac{1}{2} \Rightarrow x > 1$

Case 2: $x - 1 < 0$

$\Rightarrow x - 2 > 3x - 3$

$\Rightarrow 1 > 2x \Rightarrow x < \frac{1}{2}$

$x - 1 < 0 \ \& \ x < \frac{1}{2} \Rightarrow x < \frac{1}{2}$

Overall solution: $x < \frac{1}{2}$ or $x > 1$

Method 2: Multiplying by the square of the denominator

$(x - 2)(x - 1) < 3(x - 1)^2$

$\Rightarrow (x - 1)\big(x - 2 - 3(x - 1)\big) < 0$

$\Rightarrow (x - 1)(-2x + 1) < 0$

$\Rightarrow (x - 1)(2x - 1) > 0$

$\Rightarrow x > 1 \ \& \ x > \frac{1}{2}$ or $x < 1 \ \& \ x < \frac{1}{2}$

ie $x > 1$ or $x < \frac{1}{2}$

Method 3: Rearranging into the form $\frac{f(x)}{g(x)} < 0$ etc

$\frac{x-2}{x-1} - \frac{3(x-1)}{x-1} < 0$

$\Rightarrow \frac{-2x+1}{x-1} < 0$

$\Rightarrow \frac{2x-1}{x-1} > 0$

critical values of x: $\frac{1}{2}$ and 1

$eg\ x = 0: LHS = 1 > 0$

$eg\ x = \frac{3}{4}: LHS = \frac{\left(\frac{1}{2}\right)}{\left(-\frac{1}{4}\right)} = -2 < 0$

$eg\ \ x = 100: LHS > 0$

Solution: $x < \frac{1}{2}$ or $x > 1$

Method 4 (Graph)

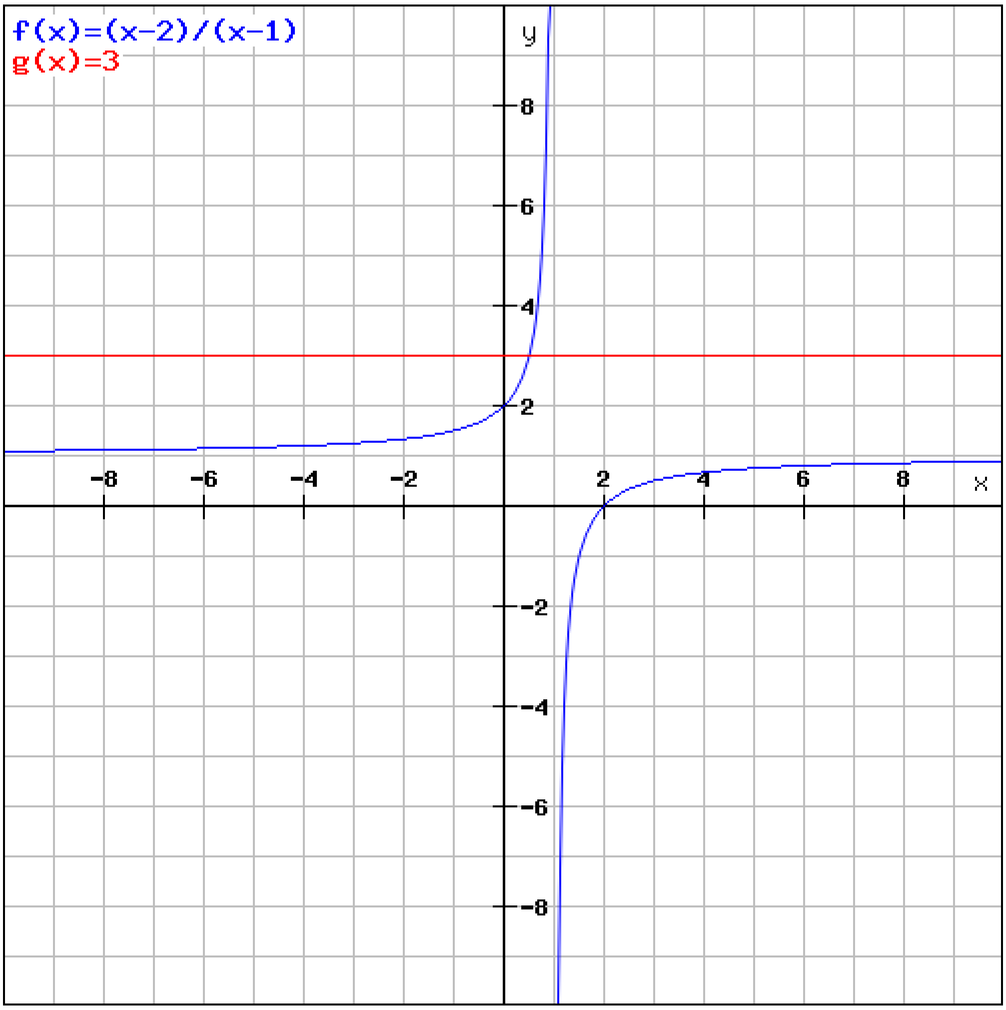
Consider the intersection of $y = \frac{x-2}{x-1}$ and $y = 3$:

$\frac{x-2}{x-1} = 3$

$\Rightarrow x - 2 = 3x - 3$

$\Rightarrow 1 = 2x \Rightarrow x = \frac{1}{2}$

$y = \frac{x-2}{x-1}$ can be sketched by noting the vertical asymptote of $x = 1$ (with $y > 0$ for $x = 1 - \delta$), and the horizontal asymptote of $y = 1$ (with $y < 1$ for eg $x = 100$)



[Noting that $\frac{x-2}{x-1} = 1 - \frac{1}{x-1}$, the graph could also be obtained from $y = \frac{1}{x}$ by the following sequence of transformations:

(i) translation of $\binom{1}{0}$, to give $y = \frac{1}{x-1}$

(ii) reflection in the x axis, to give $y = -\frac{1}{x-1}$

(iii) translation of $\binom{0}{1}$, to give $y = 1 - \frac{1}{x-1}$]

Then from the graph, $\frac{x-2}{x-1} < 3$ when $x < \frac{1}{2}$ or $x > 1$

Solve the following inequality $\frac{x}{x-1} \leq \frac{3}{x+2}$ $(x \neq 1, x \neq -2)$

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Solution

Method 1

$\frac{x}{x-1} \leq \frac{3}{x+2}$

Multiply both sides by $(x-1)^2(x+2)^2$ [as this will be positive]:

$x(x-1)(x+2)^2 \leq 3(x-1)^2(x+2)$

$\Rightarrow (x-1)(x+2)\{x(x+2) - 3(x-1)\} \leq 0$

$\Rightarrow (x-1)(x+2)(x^2 - x + 3) \leq 0$

As $x^2 - x + 3 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + 3 > 0$ for all x,

the original inequality is satisfied when $-2 \leq x \leq 1$ [for example, by considering the graph of $y = (x-1)(x+2)$].

Method 2

$\frac{x}{x-1} \leq \frac{3}{x+2} \Rightarrow \frac{x}{x-1} - \frac{3}{x+2} \leq 0$

$\Rightarrow \frac{x(x+2)-3(x-1)}{(x-1)(x+2)} \leq 0 \Rightarrow \frac{x^2-x+3}{(x-1)(x+2)} \leq 0$

As before, $x^2 - x + 3 > 0$ for all x,

so that we require $(x-1)(x+2) \leq 0$, and hence $-2 \leq x \leq 1$

Solve the inequality $|x^2 - 3| > 3x + 1$

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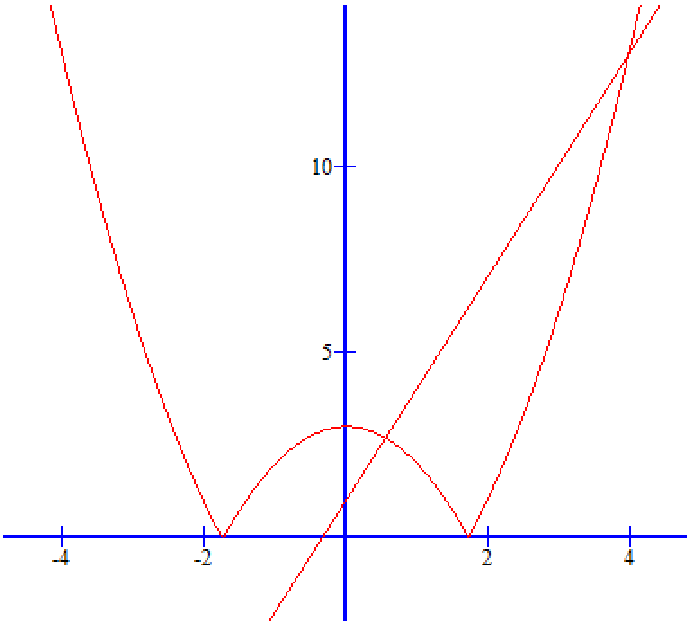
Solution



To determine the points of intersection of $y = |x^2 - 3|$ and

$y = 3x + 1$:

$x^2 - 3 \geq 0$ and $x^2 - 3 = 3x + 1 \ \Rightarrow x^2 - 3x - 4 = 0$

$\Rightarrow (x - 4)(x + 1) = 0$

From the graph, we can reject the negative value of x, to give $x = 4$, which is consistent with $x^2 - 3 \geq 0$.

Also, $x^2 - 3 < 0$ and $-(x^2 - 3) = 3x + 1 \Rightarrow x^2 + 3x - 2 = 0$

$\Rightarrow x = \frac{-3 \pm \sqrt{9+8}}{2}$

From the graph, we can reject the negative value of x, to give $\frac{-3+\sqrt{17}}{2}$, which is consistent with $x^2 - 3 < 0$.

So, from the graph, $|x^2 - 3| > 3x + 1$ when $x < \frac{-3+\sqrt{17}}{2}$ or $x > 4$

Are the following true or false?

(i) $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$

(ii) $a < b \Rightarrow a^2 < b^2$

(iii) $a < b \;\&\; c < d \Rightarrow a + c < b + d$

(iv) $a < b \;\&\; c < d \Rightarrow a - c < b - d$

Are the following true or false?

(i) $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$

(ii) $a < b \Rightarrow a^2 < b^2$

(iii) $a < b \,\&\, c < d \Rightarrow a + c < b + d$

(iv) $a < b \,\&\, c < d \Rightarrow a - c < b - d$

Solution

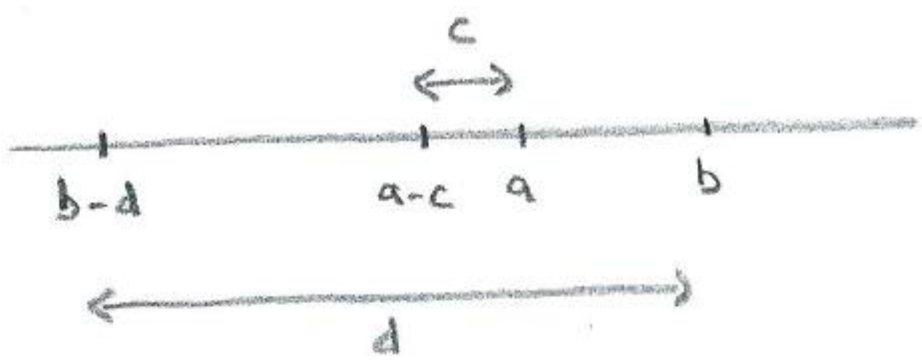
(i) Not true if $a < 0$ & $b > 0$ (consider the graph of $y = 1/x$)

(ii) Not true if $a < 0$ & $b < 0$ or

if $a < 0, b > 0$ & $|b| < |a|$(consider the graph of $y = x^2$)

(iii) True: $a < b \;\Rightarrow a + c < b + c < b + d$

(iv) False: For example, $8 < 9$ and $2 < 4$, but it is not true that $8 - 2 < 9 - 4$; see diagram



Assuming that $sin^2\theta + cos^2\theta = 1$, but without using any compound angle results, show that $sin\theta cos\theta \leq \frac{1}{2}$

Assuming that $sin^2\theta + cos^2\theta = 1$, but without using any compound angle results, show that $sin\theta cos\theta \leq \frac{1}{2}$

Solution

$(sin\theta - cos\theta)^2 \geq 0 \Rightarrow sin^2\theta + cos^2\theta - 2sin\theta cos\theta \geq 0$

$\Rightarrow 1 \geq 2sin\theta cos\theta \Rightarrow sin\theta cos\theta \leq \frac{1}{2}$

Useful result: $(\boldsymbol{a} - \boldsymbol{b})^{\mathbf{2}} \geq 0$

Is $\frac{6}{7} < \frac{2}{\sqrt{5}}$? [without using a calculator]

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Solution

$\frac{6}{7} < \frac{2}{\sqrt{5}} \Leftrightarrow \frac{36}{49} < \frac{4}{5}$

$49 \times 0.8 = \frac{1}{10}(320 + 72) = 39.2 > 36$

So $\frac{36}{49} < \frac{39.2}{49} = 0.8 = \frac{4}{5}$

Answer is Yes.

Show that $e^3 > 4e^{\frac{3}{2}}$

Show that $e^3 > 4e^{\frac{3}{2}}$

Solution

An equivalent result to prove is $e^{\frac{3}{2}} > 4$ (dividing both sides by $e^{\frac{3}{2}}$, which is positive)

$\Leftrightarrow e^3 > 16$ (as the function $y = x^2$ is increasing for $x > 0$)

$e^3 > (2 + 0.7)^3 > 2^3 + 3(2^2)(0.7) = 8 + 8.4 > 16$,

so that the original result is also true.