

Inequalities - Further Exercises (6 pages; 29/12/24)

Which is larger: $\frac{\sqrt{7}}{2}$ or $\frac{1+\sqrt{6}}{3}$ (without using a calculator)?

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Solution

Considering the difference of squares:

$$\frac{7}{4} - \frac{(1+2\sqrt{6}+6)}{9} = \frac{63-28-8\sqrt{6}}{36} > \frac{35-8(3)}{36} > 0 ; \text{ so } \frac{\sqrt{7}}{2} \text{ is larger}$$

Show that if $X > 1$ & $Y > 1$, then $X + Y < XY + 1$

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Solution

$$X + Y < XY + 1 \Leftrightarrow X + Y - XY - 1 < 0$$

$$\Leftrightarrow X(1 - Y) + Y - 1 < 0$$

$$\Leftrightarrow (X - 1)(1 - Y) < 0$$

$$\Leftrightarrow (X - 1)(Y - 1) > 0$$

$$\text{Then } X > 1 \text{ \& } Y > 1 \Rightarrow (X - 1)(Y - 1) > 0 \Rightarrow X + Y < XY + 1$$

Show that $\cos\left(\frac{3\pi}{8}\right) < \sqrt{2} - 1$

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Solution

Result to prove: $\cos\left(\frac{3\pi}{8}\right) < \sqrt{2} - 1$ (*)

Proving (*) is equivalent to proving that $\cos^2\left(\frac{3\pi}{8}\right) < (\sqrt{2} - 1)^2$,

as both sides of (*) are positive;

$$\text{or } \frac{1}{2}\left(1 + \cos\left(\frac{3\pi}{4}\right)\right) < 2 + 1 - 2\sqrt{2};$$

$$\text{or } 1 - \frac{\sqrt{2}}{2} < 6 - 4\sqrt{2};$$

$$\text{or } 10 > 7\sqrt{2};$$

which is equivalent to proving that $10^2 > 49(2)$; ie $100 > 98$

$$\text{So } \cos\left(\frac{3\pi}{8}\right) < \sqrt{2} - 1$$