Energy - Exercises (Solutions) (8 pages; 22/1/20)

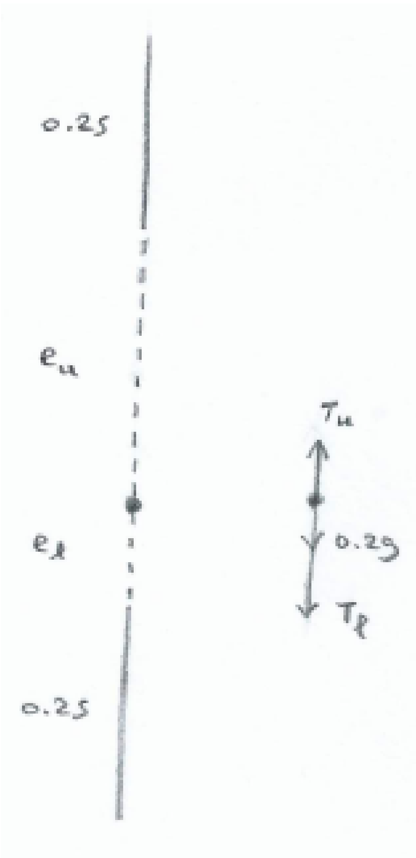
- incl. Hooke's law

(1***) A particle of mass 200g is attached at the mid-point of an elastic string of natural length 0.5m and modulus of elasticity λ, which hangs vertically between two points, 1m apart.

(i) How far will the particle be below the top point if $\lambda = 1$?

(ii) Determine the minimum value of λ such that there is no slack in the string.

Solution



(i) Let the extensions of the upper and lower parts of the string be e_u and e_l, respectively, and the tensions in the two parts T_u and T_l.

Then, referring to the diagram,

$T_u = \frac{\lambda e_u}{0.25}$, $T_l = \frac{\lambda e_l}{0.25}$ (assuming the string is not slack)

Equilibrium $\Rightarrow T_u = T_l + 0.2g$; also $e_u + e_l = 0.5$ (1)

Hence $\lambda e_u = \lambda e_l + 0.05g$

and so $\lambda e_u = \lambda(0.5 - e_u) + 0.05g$,

giving $2\lambda e_u = 0.5\lambda + 0.05g$

and hence $e_u = \frac{0.5\lambda + 0.05g}{2\lambda}$ (2)

Thus when $\lambda = 1$, $e_u = 0.495$

and the distance below the top point is $0.25 + 0.495 = 0.745m$

(ii) The string is slack if $e_l < 0$

From (1) & (2), $e_l = 0.5 - 0.25 - \frac{0.025g}{\lambda} = 0.25 - \frac{0.025g}{\lambda}$

Thus we require $0.25 - \frac{0.025g}{\lambda} \geq 0$,

so that $0.25 \geq \frac{0.025g}{\lambda}$ and $\lambda \geq 0.1g = 0.98$

(2***) A particle of mass 200g hangs at a point Q, suspended from a fixed point P, by means of a spring of original length 20cm and modulus of elasticity 5N. It is pulled down to a point R, which is 35cm below P. The particle is then released.

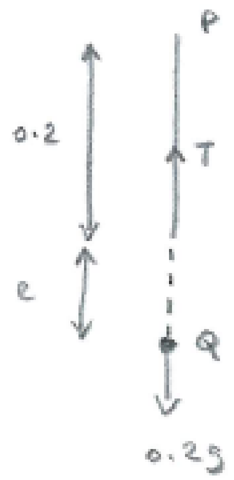
Ignoring any resistances to motion, find:

(i) the work done in pulling the particle down to R

(ii) the maximum speed of the particle after it is released, and the point at which this occurs

(iii) the distance of the particle below P when it reaches its maximum height, at position S, and show that the distance QS equals the distance QR

Solution



[Note: The g in the diagram (gravity) is not to be confused with g for grams.]

(i) If e is the extension of the spring at Q (in metres),

Hooke's Law $\Rightarrow \frac{\lambda e}{l} = T = mg \Rightarrow \frac{5e}{0.2} = (0.2)(9.8) \Rightarrow e = 0.0784$

Taking the zero of gravitational potential energy (GPE) to be R,

the total energy of the particle at Q is:

GPE + EPE + KE (where EPE is elastic potential energy & KE is kinetic energy)

$= (0.2)(9.8)(0.35 - 0.2 - 0.0784) + \frac{1}{2}\left(\frac{5}{0.2}\right)(0.0784)^2 + 0$

$= 0.140336 + 0.076832 + 0 = 0.217168$

The total energy of the particle at R is:

$0 + \frac{1}{2}\left(\frac{5}{0.2}\right)(0.15)^2 + 0 = 0.28125$

Thus the work done $= 0.28125 - 0.217168 = 0.064082 =$ 0.0641 J (3sf)

(ii) The maximum speed will occur when the particle is not accelerating; ie at Q, where the net force on the particle is zero [as $T = mg$ at the equilibrium position].

The KE of the particle at Q will equal the work done to pull it down to R, as this is the energy gained by the particle since it was last at Q.

Hence $\frac{1}{2}(0.2)v^2 = 0.064082$ (where v is the maximum speed)

and $v = 0.80051 = 0.801\ ms^{-1}$ (3sf)

(iii) Let d be the distance below P when the particle is at S.

The total energy of the particle at S is:

$(0.2)(9.8)(0.35 - d) + \frac{1}{2}\left(\frac{5}{0.2}\right)(d - 0.2)^2 + 0$

and this equals the energy at R of 0.28125 ,so that

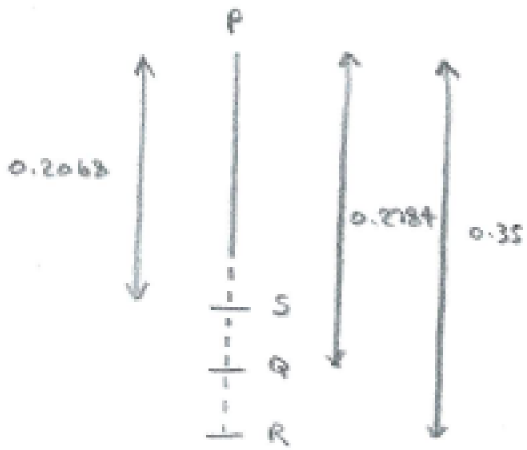
$12.5d^2 - 6.96d + 0.90475 = 0$

and $d = \frac{6.96 \pm \sqrt{3.2041}}{25} = 0.35\ or\ 0.2068$

Thus 0.35 corresponds to R and S is the point 20.68cm below P.

This is $20 + 7.84 - 20.68 = 7.16$ cm above the equilibrium position Q, whilst R is $35 - (20 + 7.84) = 7.16$ cm below Q.

[The particle oscillates between R and S.]



(3***) A bungee jumper of mass 80kg is attached to a rope of original length 10m and modulus of elasticity 1600N. How far will he or she fall? (Take g=10)

Solution

Let e be the extension of the rope.

Gain in elastic PE = loss of gravitational PE, so that

$\frac{1}{2}\left(\frac{1600}{10}\right)e^2 = 80(10)(10 + e)$

$\Rightarrow e^2 = 100 + 10e$

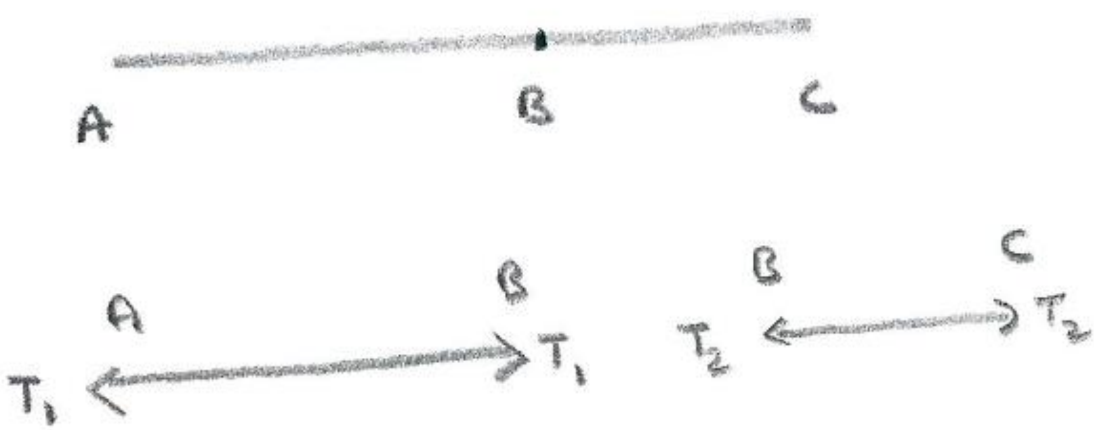
$\Rightarrow e^2 - 10e - 100 = 0$

$\Rightarrow e = \frac{10 \pm \sqrt{100+400}}{2} = 16.18$m (ignoring -ve value)

So bungee jumper falls by $10 + 16.18 = 26.18m$

(4***) Two elastic strings AB and BC are joined together at B, to form one long string. String AB has natural length $4m$ and modulus of elasticity $20N$; string BC has natural length $2m$ and modulus of elasticity $30N$. The ends A and C of the long string are attached to two fixed points which are $10m$ apart. Find the tension in the combined string.

Solution



Considering the force diagram for AB: by N2L, the reaction at A will equal the force applied by string BC at B. Call this T_1 - this is the tension that AB is under. Similarly, string BC will be under tension T_2. By N3L, the forces that the two strings apply to each other will be equal and opposite, so that $T_1 = T_2 = T$, say. The tension in the combined string (determined by the reactions at A and C) will therefore be T also.

For string AB, Hooke's law $\Rightarrow T_1 = \frac{20e_1}{4}$, where e_1 is the extension of string AB.

Similarly, for string BC, $T_2 = \frac{30e_2}{2}$

Also $(4 + e_1) + (2 + e_2) = 10$, so that $e_1 + e_2 = 4$

Then $T_1 = T_2 \Rightarrow \frac{20e_1}{4} = \frac{30e_2}{2}$, so that $5e_1 = 15(4 - e_1)$,

and $e_1 = 3(4 - e_1)$, so that $4e_1 = 12$ and hence $e_1 = 3$, and $e_2 = 1$.

Therefore $T = 5e_1 = 15N$

Note: A result [which would probably have to be proved in an exam, though possibly not for STEP] that can be applied here is that, with strings of stiffness k_1 and k_2 in series, the combined string has stiffness k given by $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$

In this case, $\frac{1}{k} = \frac{4}{20} + \frac{2}{30} = \frac{16}{60}$, so that $k = \frac{15}{4}$,

and $T = k(e_1 + e_2) = \frac{15}{4}(4) = 15N$

(5***) A car of mass 1 tonne starts to climb a hill at $20ms^{-1}$. The slope of the hill is a constant θ, where $sin\theta = \frac{1}{10}$. If the car is not accelerating (or braking) and there is a constant resistance to motion of $1000N$, find the speed of the car when it has gained a height of $5m$. Assume that $g = 10$.

Solution

Method 1

By the Work-Energy principle,

Gain in KE = Work done by forces,

so that $\frac{1}{2}(1000)(v^2 - 20^2) = -1000g(5) - 1000\left(\frac{5}{sin\theta}\right)$

$\Rightarrow 500v^2 = 200000 - 50000 - 50000$

$\Rightarrow v^2 = 200 \Rightarrow v = 14.1\ ms^{-1}\ (3sf)$

Method 2

By Conservation of Energy,

Gain in PE $=$ loss of KE $-$ work done against resistance

$\Rightarrow 1000g(5) = \frac{1}{2}(1000)(20^2 - v^2) - 1000\left(\frac{5}{sin\theta}\right)$

which gives the same equation.