Ellipses - Exercises (Solutions) (3 pages; 30/12/19)

(1**) Show that the equation of the tangent to the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1,y_1) is $\frac{yy_1}{b^2}+\frac{xx_1}{a^2}=1$

Solution

Differentiating gives $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$, so that $\frac{dy}{dx}=-\frac{xb^2}{ya^2}$

Then the tangent at (x_1,y_1) is $\frac{y-y_1}{x-x_1}=-\frac{x_1b^2}{y_1a^2}$,

or $yy_1a^2-{y_1}^2a^2=-xx_1b^2+{x_1}^2b^2$

and hence $\frac{yy_1}{b^2}-\frac{{y_1}^2}{b^2}=-\frac{xx_1}{a^2}+\frac{{x_1}^2}{a^2}$

or $\frac{yy_1}{b^2}+\frac{xx_1}{a^2}=\frac{{x_1}^2}{a^2}+\frac{{y_1}^2}{b^2}$

As (x_1,y_1) lies on the ellipse, $\frac{{x_1}^2}{a^2}+\frac{{y_1}^2}{b^2}=1$,

so we have $\frac{yy_1}{b^2}+\frac{xx_1}{a^2}=1$ as the equation of the tangent.

(2***) Given the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and circle $x^2+y^2=a^2$, let l_1 be the tangent to the ellipse at the point $(acos\theta, bsin\theta)$ and l_2 be the tangent to the circle at the point $(acos\theta, asin\theta)$. Find the locus of the point of intersection of l_1 & l_2, as θ varies.

Solution

The equation of l_1 is $\frac{y-bsin\theta}{x-acos\theta}=\frac{dy/d\theta}{dx/d\theta}=\frac{bcos\theta}{-asin\theta}$ (1)

The equation of l_2 is $\frac{y-asin\theta}{x-acos\theta}=\frac{dy/d\theta}{dx/d\theta}=\frac{acos\theta}{-asin\theta}$ (2)

At the intersection of l_1 & l_2,

$x - acos\theta = \frac{-asin\theta}{bcos\theta}(y - bsin\theta)$ from (1)

and $x - acos\theta = \frac{-sin\theta}{cos\theta}(y - asin\theta)$ from (2),

so that $\left(\frac{a}{b}\right)(y - bsin\theta) = y - asin\theta$

$\Rightarrow ay - absin\theta = by - absin\theta$

$\Rightarrow y = 0,$ as $a \neq b$ (otherwise the ellipse would be a circle)

Then, from (2), $x - acos\theta = \frac{asin^2\theta}{cos\theta}$,

so that $xcos\theta = acos^2\theta + asin^2\theta = a$, and thus $x = \frac{a}{cos\theta}$

As $-1 < cos\theta < 1$, x can take values in the range
$(-\infty, -a]$ & $[a, \infty)$

Thus the required locus is the set of points on the x-axis in the above range.

(3***) Show that the area within the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab

Solution

Area $= 4\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}\, dx$

Let $x = asin\theta$, so that $dx = acos\theta\, d\theta$

Then Area $= 4b\int_0^{\frac{\pi}{2}} cos\theta . acos\theta\, d\theta$

$= 2ab\int_0^{\frac{\pi}{2}} 1 + cos2\theta\, d\theta$

$$= 2ab\left[\theta + \frac{1}{2}sin2\theta\right]_0^{\frac{\pi}{2}}$$

$$= 2ab\left(\frac{\pi}{2}\right) = \pi ab$$