Eigenvectors (10 pages; 4/9/18)

(1) Finding eigenvalues & eigenvectors

(i) Introduction

Eigenvectors are invariant lines through the Origin; ie where a transformation maps a point onto another point on the line.

If $T\underline{x} = \lambda\underline{x}$ then $\underline{x}$ is termed an eigenvector of the matrix T, with λ being the eigenvalue.

Interpreting $\underline{x}$ as a position vector (ie anchored at the origin),

$\lambda\underline{x}$ has the same direction as $\underline{x}$; $\underline{x}$ has been magnified by λ

It is assumed that $\underline{x}$ is non-zero.

If the eigenvalue is $\lambda = 1$, then we have found a line of invariant points.

(ii) **Example**

To find the eigenvalues and eigenvectors for $T = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

Step 1: Find the eigenvalues

$T\underline{x} = \lambda\underline{x} = \lambda I\underline{x} \Rightarrow (T - \lambda I)\underline{x} = 0$

We require there to be more than one solution to this equation ($\underline{x} = 0$ and a non-zero solution). Hence $|T - \lambda I| = 0$

ie $\begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = 0$ (the 'characteristic equation')

$\Rightarrow (1-\lambda)(1-\lambda)(1-\lambda) - 0 - 1(1-\lambda) = 0$

$$\Rightarrow -\lambda^3 + 3\lambda^2 - 3\lambda + 1 - 1 + \lambda = 0$$

$$\Rightarrow \lambda(\lambda^2 - 3\lambda + 2) = 0$$

$$\Rightarrow \lambda(\lambda - 1)(\lambda - 2) = 0$$

$$\Rightarrow \lambda = 0,\ 1 \text{ or } 2$$

Step 2: Find the corresponding eigenvectors

$$\begin{pmatrix} 1-\lambda & 1 & -1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda = 0 \Rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow y = 0; x - z = 0$$

$$\text{eg} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Notes

(a) Any multiple of this vector is an eigenvector with eigenvalue 0.

(b) Alternative layout: $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

The eigenvector corresponding to $\lambda = 1$ is given by:

$$\begin{pmatrix} 1-1 & 1 & -1 \\ 0 & 1-1 & 0 \\ -1 & 0 & 1-1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow y - z = 0; -x = 0$$

$\Rightarrow$ eigenvector of $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

The eigenvector corresponding to $\lambda = 2$ is given by:

$$\begin{pmatrix} 1-2 & 1 & -1 \\ 0 & 1-2 & 0 \\ -1 & 0 & 1-2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -\mathrm{x} + \mathrm{y} - \mathrm{z} = 0; -\mathrm{y} = 0 \ \Rightarrow$ eigenvector of $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Note that the last equation $[-x - z = 0]$ is consistent with the first two; ie finding the eigenvectors provides an automatic check on the eigenvalues.

Thus the eigenvectors of T are $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

(with eigenvalues of 0, 1 & 2 respectively)

Note that $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ could be written as $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ instead, for example.

In general, $\begin{pmatrix} a \\ -b \\ -c \end{pmatrix}$ is sometimes preferred to $\begin{pmatrix} -a \\ b \\ c \end{pmatrix}$ (where a, b & c are positive).

So, for this example, any position vector of the form $k\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is transformed to $2k\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$k\begin{pmatrix}1\\0\\-1\end{pmatrix}$ represents an **invariant line** (note that the points on the line transform to **other** points on the line);

Its cartesian equation is found as follows:

$\begin{pmatrix}x\\y\\z\end{pmatrix} = k\begin{pmatrix}1\\0\\-1\end{pmatrix}$, so that $k = x = -z$

ie the cartesian equation is $x = -z, y = 0$

Note: There may be other invariant lines, not passing through the origin.

As position vectors of the form $k\begin{pmatrix}0\\1\\1\end{pmatrix}$ transform to $k\begin{pmatrix}0\\1\\1\end{pmatrix}$, this represents a **line of invariant points** (where the eigenvalue is 1), with cartesian equation $y = z, x = 0$ (from the earlier equations).

(iii) **Example**: Find the eigenvalues & eigenvectors of the matrix $\begin{pmatrix}4 & 2\\1 & 3\end{pmatrix}$

Solution: $\begin{vmatrix}4-\lambda & 2\\1 & 3-\lambda\end{vmatrix} = 0$

$\Rightarrow (4-\lambda)(3-\lambda) - 2 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 10 = 0$

$\Rightarrow (\lambda - 5)(\lambda - 2) = 0$

$\Rightarrow \lambda = 2\ or\ 5$

$\lambda = 2 \Rightarrow \begin{pmatrix}2 & 2\\1 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow x + y = 0 \Rightarrow$ eigenvector of $\begin{pmatrix}1\\-1\end{pmatrix}$

$\lambda = 5 \Rightarrow \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -x + 2y = 0$

$\Rightarrow$ eigenvector of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The equations of the invariant lines representing these eigenvectors are: $y = -x$ and $y = \frac{1}{2}x$

(2) Diagonalisation of a (general) matrix

(i) A matrix M is said to be diagonalisable if there exists a diagonal matrix D such that $M = PDP^{-1}$.

Matrices A and B are said to be similar if $B = PAP^{-1}$ for some matrix P, but there is no requirement for A (or B) to be diagonal. As you've probably gathered, there's no shortage of definitions in the subject of matrices.

Thus, a matrix is diagonalisable if it is similar to a diagonal matrix.

(ii) We therefore need to find an invertible matrix P such that

$MP = PD$

For the matrix $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$, we saw that the eigenvectors were

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, with corresponding eigenvalues of 2 and 5

Then $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 5\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

so that $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$

[Note that the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under the transformation $\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so that the image of $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is $2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$]

This has the required form $MP = PD$, so that the columns of P are the eigenvectors of M and the diagonal elements of D are the eigenvalues of M (in the same order as the corresponding eigenvectors appear in P).

(iii) Diagonalisability

In order for an $n \times n$ matrix M to be diagonalisable, it must have n linearly independent eigenvectors, so that $\det P \neq 0$, and hence P is invertible. Thus it is necessary for M to have only real eigenvalues.

If the eigenvalues are real and distinct, then the eigenvectors will be linearly independent.

If there are repeated eigenvalues, then it is still possible for the eigenvectors to be linearly independent - see Part 4, (F)(10).

(iv) The diagonalising of M can be associated with a coordinate transformation, as follows:

If $M = PDP^{-1}$, it will have the same eigenvalues as D. For the example above,

$\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 5\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

[again, noting that the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under the transformation $\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$ is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$]

Effectively the eigenvectors have been transformed to the simpler coordinates $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Writing $T\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$ and $T\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$, so that T is the matrix representing this coordinate transformation, we see that

$T\begin{pmatrix}1&2\\-1&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$; ie $TP=I$, and hence $T=P^{-1}$

Also, if $\begin{pmatrix}a\\b\end{pmatrix}$ is a general vector in the original coordinate system, and if it can be expressed in terms of the eigenvectors as

$\alpha\begin{pmatrix}1\\-1\end{pmatrix}+\beta\begin{pmatrix}2\\1\end{pmatrix}$, then $\begin{pmatrix}a\\b\end{pmatrix}$ will be transformed to the new coordinates $P^{-1}\begin{pmatrix}a\\b\end{pmatrix}=\alpha P^{-1}\begin{pmatrix}1\\-1\end{pmatrix}+\beta P^{-1}\begin{pmatrix}2\\1\end{pmatrix}$

$=\alpha\begin{pmatrix}1\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\alpha\\\beta\end{pmatrix}$

Thus, in the old system, the image of $\begin{pmatrix}a\\b\end{pmatrix}$ under M is

$M\begin{pmatrix}a\\b\end{pmatrix}=PDP^{-1}\begin{pmatrix}a\\b\end{pmatrix}$, whereas after transforming coordinates, the image of $\begin{pmatrix}\alpha\\\beta\end{pmatrix}$ under D is just $D\begin{pmatrix}\alpha\\\beta\end{pmatrix}$

Note also that $PDP^{-1}\begin{pmatrix}a\\b\end{pmatrix}$ can be interpreted as follows:

First apply the coordinate transformation P^{-1} to $\begin{pmatrix}a\\b\end{pmatrix}$, to obtain $P^{-1}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}\alpha\\\beta\end{pmatrix}$; then apply D in the new system, to give $D\begin{pmatrix}\alpha\\\beta\end{pmatrix}$; and finally convert back to the old system by applying the inverse of P^{-1}, to give $PD\begin{pmatrix}\alpha\\\beta\end{pmatrix}$.

[The change of coordinates described above involves changing the basis of $\mathbb{R}^2$ from $\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}$ to $\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}$]

(v) If $M=PDP^{-1}$, then

$M^n = (PDP^{-1})(PDP^{-1})(PDP^{-1})\ldots = PDIDID\ldots IDP^{-1}$

$= PD^nP^{-1}$

(vi) If $M = PDP^{-1}$, then $|M| = |P|.|D|.|P^{-1}| = |P|.|P^{-1}|.|D|$

$= |PP^{-1}|.|D| = |I|.|D| = |D|$

(vii) Even if a matrix M isn't diagonalisable, it is generally possible to find a triangular matrix T * that is similar to M; ie such that $M = PTP^{-1}$. If M has eigenvalues, then these will be shared by T.

* (upper) triangular matrix: eg $\begin{pmatrix} 2 & 3 & 5 & 8 \\ 0 & 4 & 6 & 9 \\ 0 & 0 & 7 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

(where the entries below the diagonal are zero, and the diagonal entries could be zero)

(3) Diagonalisation of a symmetric matrix

(i) Preliminary definitions and results

(a) A matrix A is described as orthogonal if $A^{-1} = A^T$

(b) It can be shown that a matrix is orthogonal if and only if

1. its columns are mutually orthogonal (ie perpendicular, so that their scalar product is zero)
2. each column has unit magnitude

[See Matrices - Exercises (Part 2) for a proof.]

(c) An eigenvector is 'normalised', by dividing each of its elements by its magnitude. It then has unit magnitude.

For example, $\begin{pmatrix}1\\2\\3\end{pmatrix}$ becomes $\frac{1}{\sqrt{1+4+9}}\begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}$

(ii) If M is symmetric, then the following results can be established:

(a) M has only real eigenvalues [the proof involves matrices with complex elements].

(b) M can always be diagonalised (ie whether the eigenvalues are distinct or not).

(c) The eigenvectors of M will be mutually orthogonal.

Proof

First of all, we note that the scalar product of two eigenvectors $\underline{x}_1$ & $\underline{x}_2$ is $\underline{x}_1{}^T\underline{x}_2$ (or $\underline{x}_2{}^T\underline{x}_1$); ie a row vector multiplied by a column vector.

Suppose then that $M\underline{x}_1 = \lambda_1\underline{x}_1$ & $M\underline{x}_2 = \lambda_2\underline{x}_2$, with M being symmetric.

Taking the transpose of both sides of the first equation gives

$\underline{x}_1{}^T M^T = \lambda_1\underline{x}_1{}^T$ or $\underline{x}_1{}^T M = \lambda_1\underline{x}_1{}^T$, as M is symmetric

Multiplying on the right by $\underline{x}_2$ (which is compatible) then gives

$\underline{x}_1{}^T M\underline{x}_2 = \lambda_1\underline{x}_1{}^T\underline{x}_2$ and hence $\underline{x}_1{}^T\lambda_2\underline{x}_2 = \lambda_1\underline{x}_1{}^T\underline{x}_2$

Rearranging, we have $(\lambda_2 - \lambda_1)\underline{x}_1{}^T\underline{x}_2 = 0$,

so that, assuming $\lambda_1 \neq \lambda_2$, $\underline{x}_1{}^T\underline{x}_2 = 0$ and thus the eigenvectors are perpendicular to each other, or 'mutually orthogonal'.

(d) If the eigenvectors of M are 'normalised' , then P will be orthogonal (ie $P^{-1} = P^T$).

Suppose that Q is the matrix of eigenvectors prior to normalisation, and that P is the matrix of normalised eigenvectors.

Then $M = QDQ^{-1} = PDP^{-1} = PDP^T$

(note that the change from P to Q is balanced by the change from P^{-1} to Q^{-1})

P^T is of course much easier to determine than P^{-1}

[By definition, the eigenvectors then form an orthonormal set of vectors - as a result of being mutually orthogonal and of unit magnitude.]

(iii) When P is orthogonal, M is said to be orthogonally diagonalisable, and it can be shown that this is the case if and only if M is symmetric [see Matrices - Exercises (Part 2) for the easier half of the proof].

(4) Miscellaneous

(i) The sum of the eigenvalues of M equals the trace of M (the sum of the elements on the main diagonal). [See Matrices - Exercises (Part 2)]

(ii) The product of the eigenvalues of M equals $\det M$. [See Matrices - Exercises (Part 2)]