Discrete Random Variables – Exercises

(10 pages; 14/2/26)

(1) The probability distribution of a discrete random variable is shown below.

x	1	2	3
$P(X = x)$	p	q	r

Given that $E(X) = \frac{7}{3}$ and $Var(X) = \frac{5}{9}$, find p, q and r.

(1) The probability distribution of a discrete random variable is shown below.

x	1	2	3
$P(X=x)$	p	q	r

Given that $E(X)=\frac{7}{3}$ and $Var(X)=\frac{5}{9}$, find p, q and r.

Solution

$p+q+r=1$ (1)

$E(X)=\frac{7}{3} \Rightarrow 1(p)+2(q)+3(r)=\frac{7}{3}$; $3p+6q+9r=7$ (2)

$Var(X)=\frac{5}{9} \Rightarrow E(X^2)-[E(X)]^2=\frac{5}{9}$

$\Rightarrow 1(p)+4(q)+9(r)-\frac{49}{9}=\frac{5}{9}$

$\Rightarrow p+4q+9r=6$ (3)

Using (1) to eliminate r from (2) & (3):

$3p+6q+9(1-\mathrm{p}-\mathrm{q})=7$; $-6p-3q=-2$; $6p+3q=2$ (4)

and $p+4q+9(1-\mathrm{p}-\mathrm{q})=6$; $-8p-5q=-3$;

$8p+5q=3$ (5)

Then $4(4)-3(5) \Rightarrow -3q=-1; q=\frac{1}{3}$

and (4) $\Rightarrow 6p=2-1; p=\frac{1}{6}$

and (1) $\Rightarrow r=1-\frac{1}{6}-\frac{1}{3}=\frac{1}{2}$

(2) The random variables X_i , for $i = 1$ to 100, are independent, and $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$

Find:

(i) $Var(X_1)$

(ii) $Var(X_1 + X_2 + \cdots + X_{100})$

(iii) $Var(100X_1)$

(iv) $Var(X_1 - X_2)$

(2) The random variables X_i , for $i = 1$ to 100, are independent, and $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$

Find:

(i) $Var(X_1)$

(ii) $Var(X_1 + X_2 + \cdots + X_{100})$

(iii) $Var(100X_1)$

(iv) $Var(X_1 - X_2)$

Solution

(i) $Var(X_1) = E\left({X_1}^2\right) - (E(X_1))^2$

$= \left\{\frac{1}{2} .1 + \frac{1}{2} . 1\right\} - 0 = 1$

(ii) $Var(X_1 + X_2 + \cdots + X_{100})$

$= Var(X_1) + Var(X_2) + \cdots + Var(X_{100})$

(as the X_i are independent)

$= 100(1) = 100$

(iii) $Var(100X_1) = 100^2 Var(X_1) = 10000$

(iv) $Var(X_1 - X_2) = 1^2 Var(X_1) + (-1)^2 Var(X_2)$

$= 1 + 1 = 2$

(3)(i) Show algebraically that $E[aX+b]=aE(X)+b$

(ii) Show that $Var(X)=E(X^2)-\mu^2$

(iii) Show that $Var(aX+b)=a^2VarX$

(3)(i) Show algebraically that $E[aX+b]=aE(X)+b$

(ii) Show that $Var(X)=E(X^2)-\mu^2$

(iii) Show that $Var(aX+b)=a^2VarX$

Solution

(i) $E[aX+b]=\sum_x(ax+b)P(X=x)$

$=[a\sum_x xP(X=x)]+b\sum_x P(X=x)$

$=aE(X)+b$

(ii) $Var(X)=E[(X-\mu)^2]$

$=E[X^2-2X\mu+\mu^2]$

$=\sum_x(x^2-2x\mu+\mu^2)P(X=x)$

$=[\sum_x x^2P(X=x)]-[2\mu\sum_x xP(X=x)]+\mu^2\sum_x P(X=x)$

$=E(X^2)-2\mu E(X)+\ \mu^2$

$=E(X^2)-2\mu^2+\mu^2$

$=E(X^2)-\mu^2$

(iii) $Var(aX+b)=E[(aX+b)^2]-[E(aX+b)]^2$

$=E[a^2X^2+2abX+b^2]-[aE(X)+b]^2$

$=a^2E(X^2)+2abE(X)+b^2-[a^2[E(X)]^2+2abE(X)+b^2]$

$=a^2E(X^2)-a^2[E(X)]^2$

$=a^2VarX$

(4) Show that $\sum_{x=0}^{n} P(X = x) = 1$ for the Binomial distribution.

(4) Show that $\sum_{x=0}^{n} P(X = x) = 1$ for the Binomial distribution.

Solution

$\sum_{x=0}^{n} P(X = x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$

$= (p + [1-p])^n = 1$

(5) If $X \sim B(n, p)$, prove that $E(X) = np$

(5) If $X \sim B(n,p)$, prove that $E(X) = np$

Solution

$E(X) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} x$

$= \sum_{x=1}^{n} \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} x$ (as the 1st term vanishes)

$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{(x-1)} (1-p)^{n-x}$

$= np \sum_{x-1=0}^{n-1} \frac{(n-1)!}{(x-1)!(n-x)!} p^{(x-1)} (1-p)^{n-x}$

Let $u = x - 1$ and $N = n - 1$

Then $E(X) = np \sum_{u=0}^{N} \frac{N!}{u!(N-u)!} p^u (1-p)^{N-u}$

$= np \sum_u P(X = u) = np$