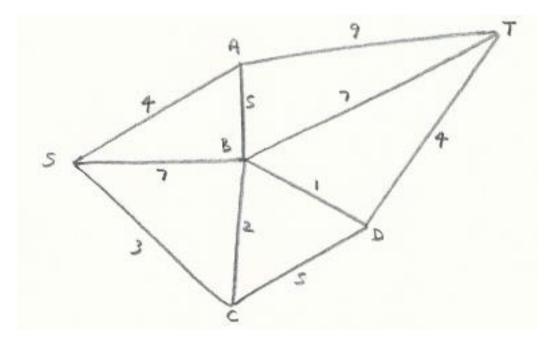# Dijkstra's Algorithm (5/9/2013)

(Shortest Path Problem)

The aim is to find the shortest path between two specified nodes.

The idea with this algorithm is to attach to each node a label showing its shortest distance from the start node (ie using the optimum route).
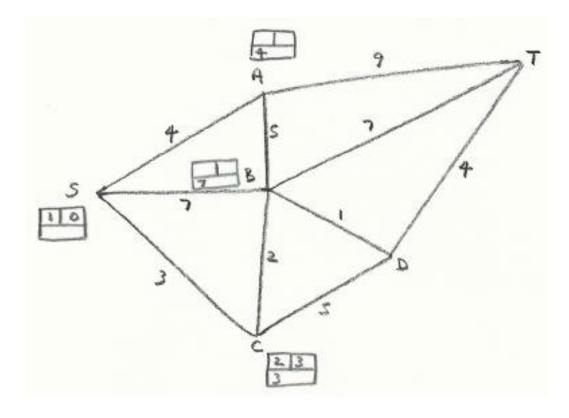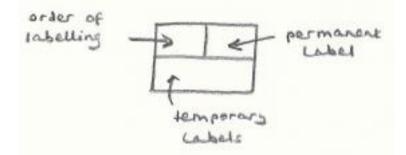
Example

(starting at S and finishing at T)

We start by attaching 'temporary' labels to A, B and C of 4, 7 and 3 respectively (these being the distances directly from S). The shortest distance from S to C is then clearly 3 (since an indirect route via A or B has to produce a distance greater than 3). Thus C can be given a 'permanent' label for the shortest distance. The 'temporary' labels of A and B will be modified later if a shorter (indirect) route is subsequently discovered.
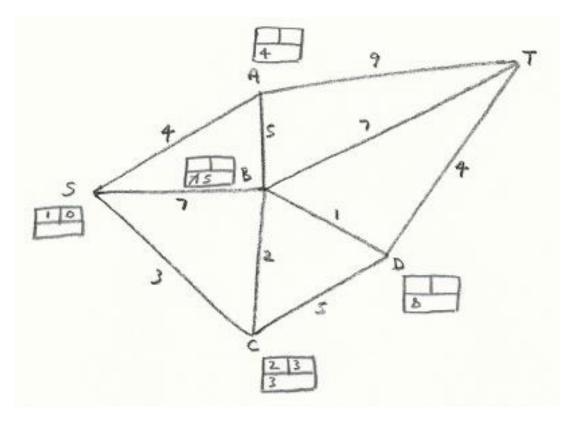
As can be seen above, each node has a box attached to it. The labelling convention is as follows:

We now consider the nodes that are immediate neighbours of C (the one that has just had a permanent label assigned to it); namely B and D. In the case of B, we can improve on the temporary label of 7, since the route via C will have distance 3+2=5. Thus the temporary label is changed to 5.

The shortest distance from S to D, via C, is 3+5=8, and so D is given the temporary label 8.

We now consider the node with the smallest temporary label; in this case, A - with 4. This will, in fact, be the shortest distance from S.
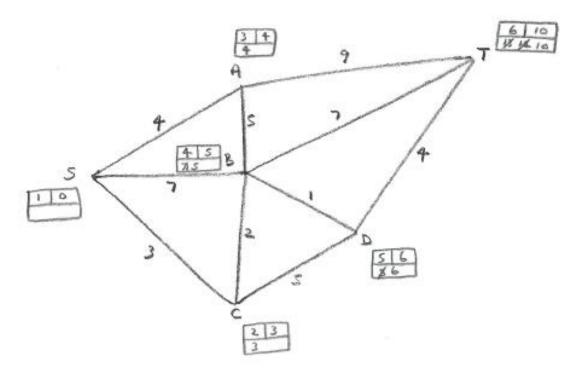
[If you're interested, the reasoning behind this is as follows:

We are trying to establish that the shortest route from S to A is just SA (the route that gives rise to the temporary label of 4.

Suppose that the shortest route is some other path(eg SCBA), so that the first temporary label that is reached is not A (eg B). Because this is the shortest route to A, it will also be the shortest route to (eg) B, which means that B's temporary label of 5 must be its permanent label, since the value of 5 was arrived at by considering all routes directly from a permanent label - including C. But this leads to a contradiction, since we know that the shortest route to A has a length no greater than 4, and we have already taken 5 to reach B. Hence our supposition was incorrect: the first temporary label that is reached is in fact A, and SA is the shortest route to A.]

Note: If more than one node shares the smallest temporary label, it doesn't matter which is chosen.

From now on we will refer to the final diagram, shown below. This is the only diagram that is required when answering exam questions. The boxes will make it clear that the algorithm has been applied correctly.

The procedure from now on is simply a repeat of what we have done so far, but will be described, in order to reinforce the ideas involved.

We now consider the nodes that can be reached directly from A; namely B and T. In the case of B, we can't improve on the temporary label of 5, since the route directly from A will have distance 4+5=9. The shortest distance from S to T directly from A is 4+9=13, and so T is given the temporary label 13.

We once again consider the node with the smallest temporary label; in this case, B with 5. This will be the shortest distance from S, since we have already investigated the routes via the permanently labelled A and C (C did in fact generate the label of 5), and routes via the temporarily labelled nodes D and T can once again be rejected (as B has been chosen because it has the smallest temporary label). Thus B's label is made permanent.

As can be seen, we alternate between (i) selecting the smallest temporary label and making it permanent and (ii) labelling (or re-labelling) its neighbouring nodes with temporary labels.

Thus we now examine the nodes with temporary labels neighbouring B; ie D and T. For D, the label of 8 can be improved on, since the shortest route directly from B has distance 5+1=6. For T, the label of 13 can also be improved on, since the shortest route directly from B has distance 5+7=12.

The node with the smallest temporary label is now D, and this is made permanent.

Finally, we consider the route to T directly from D: the shortest distance along this route is 6+4=10, which is an improvement on the temporary label of 12.

As T is now the only remaining temporary label, it is therefore the smallest temporary label, and can safely be made permanent.

The shortest distance from S to T is thus 10. The shortest route is determined by looking for neighbouring nodes for which the length of the arc between them equals the difference between their permanent labels.

In this example, working backwards from T, the last stage has to be DT (since 10-6=4, which is the arc length).

Then BD=1 is the next leg, followed by CB=2, and finally SC=3.

Thus the shortest route is SCBDT, with a total distance of 3+2+1+4=10 (which – as a useful check - equals the label of T).