Numerical methods: Differential Equations

Exercises (with Sol'ns) (6 pages; 8/3/26)

(1) The function $y(x)$ satisfies the differential equation

$\frac{d^2y}{dx^2} = \frac{ye^x}{x+1}$, subject to the initial conditions $y = 2$ and $\frac{dy}{dx} = 1$ when $x = 0$.

Use 2 iterations of the approximations $\left(\frac{d^2y}{dx^2}\right)_{n+1} \approx \frac{y_n + y_{n+2} - 2y_{n+1}}{h^2}$

and $\left(\frac{dy}{dx}\right)_{n+1} \approx \frac{y_{n+2} - y_n}{2h}$ to estimate the value of y when $x = 0.4$

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Solution

[Note: The 2nd Order method indicated in the question can be recalled by writing $\left(\frac{d^2y}{dx^2}\right)_{n+1} \approx \frac{(y_{n+2} - y_{n+1}) - (y_{n+1} - y_n)}{h^2}$]

We can set up a table (with $h = 0.2$), with the initial conditions:

i	x_i	y_i	$\left(\frac{dy}{dx}\right)_i$	$\left(\frac{d^2y}{dx^2}\right)_i$
0	0	2	1	
1	0.2			
2	0.4			

$\left(\frac{d^2y}{dx^2}\right)_0 = \frac{y_0 e^{x_0}}{x_0 + 1} = \frac{2e^0}{0+1} = 2$

Also, $1 = \left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_{-1}}{2h} = \frac{y_1 - y_{-1}}{0.4}$ (1)

and $2 = \left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_{-1} + y_1 - 2y_0}{h^2} = \frac{y_{-1} + y_1 - 4}{0.04}$ (2)

[only y_1 is actually needed though]

Eliminating y_{-1} from (1) and (2),

$y_{-1} = y_1 - 0.4$, from (1),

and then (2) gives $2 = \frac{2y_1 - 4.4}{0.04}$, so that $y_1 = \frac{1}{2}(0.08 + 4.4) = 2.24$

i	x_i	y_i	$\left(\frac{dy}{dx}\right)_i$	$\left(\frac{d^2y}{dx^2}\right)_i$
0	0	2	1	2
1	0.2	2.24		
2	0.4			

For the next iteration,

$$\left(\frac{d^2y}{dx^2}\right)_1 = \frac{y_1 e^{x_1}}{x_1+1} = \frac{2.24e^{0.2}}{0.2+1} = 2.27995$$

and also $\left(\frac{d^2y}{dx^2}\right)_1 \approx \frac{y_0+y_2-2y_1}{h^2} = \frac{2+y_2-2(2.24)}{0.04}$,

and so $2.27995 = \frac{y_2-2.48}{0.04}$,

and $y_2 = 2.27995(0.04) + 2.48 = 2.57120$ or 2.57 to 2dp

(2) The function $y(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} = x^2 + 2y,$$

subject to the boundary conditions $y = 2$ and $\frac{dy}{dx} = 4$ when $x = 1$.

Use 2 iterations of the approximations $\left(\frac{d^2y}{dx^2}\right)_{n+1} \approx \frac{y_n + y_{n+2} - 2y_{n+1}}{h^2}$

and $\left(\frac{dy}{dx}\right)_{n+1} \approx \frac{y_{n+2} - y_n}{2h}$ to estimate the value of y when $x = 1.2$

(2) The function $y(x)$ satisfies the differential equation

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and $\left(\frac{dy}{dx}\right)_{n+1} \approx \frac{y_{n+2} - y_n}{2h}$ to estimate the value of y when $x = 1.2$

Solution

We can set up a table (with $h = 0.1$), with the boundary conditions:

i	x_i	y_i	$\left(\frac{dy}{dx}\right)_i$	$\left(\frac{d^2y}{dx^2}\right)_i$
0	1	2	4	
1	1.1			
2	1.2			

$$\left(\frac{d^2y}{dx^2}\right)_0 - 3x_0\left(\frac{dy}{dx}\right)_0 = {x_0}^2 + 2y_0,$$

so that $\left(\frac{d^2y}{dx^2}\right)_0 - 3(1)(4) = 1^2 + 2(2)$; $\left(\frac{d^2y}{dx^2}\right)_0 = 17$

Also, $4 = \left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_{-1}}{2h} = \frac{y_1 - y_{-1}}{0.2}$ (1)

and $17 = \left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_{-1} + y_1 - 2y_0}{h^2} = \frac{y_{-1} + y_1 - 4}{0.01}$ (2)

[only y_1 is actually needed though]

Eliminating y_{-1} from (1) and (2),

$y_{-1} = y_1 - 0.8$, from (1),

and then (2) gives $17 = \frac{2y_1 - 4.8}{0.01}$, so that $y_1 = \frac{1}{2}(0.17 + 4.8) = 2.485$

i	x_i	y_i	$\left(\frac{dy}{dx}\right)_i$	$\left(\frac{d^2y}{dx^2}\right)_i$
0	1	2	4	17
1	1.1	2.485		
2	1.2			

For the next iteration,

$\left(\frac{d^2y}{dx^2}\right)_1 - 3x_1\left(\frac{dy}{dx}\right)_1 = {x_1}^2 + 2y_1,$

so that $\left(\frac{d^2y}{dx^2}\right)_1 - 3(1.1)\left(\frac{dy}{dx}\right)_1 = (1.1)^2 + 2(2.485)$

Also, $\left(\frac{dy}{dx}\right)_1 \approx \frac{y_2 - y_0}{2h} = \frac{y_2 - 2}{0.2}$

and $\left(\frac{d^2y}{dx^2}\right)_1 \approx \frac{y_0 + y_2 - 2y_1}{h^2} = \frac{2 + y_2 - 2(2.485)}{0.01}$,

so that $\frac{2 + y_2 - 2(2.485)}{0.01} - 3(1.1)\left(\frac{y_2 - 2}{0.2}\right) = (1.1)^2 + 2(2.485),$

and hence $\frac{y_2 - 2.97}{0.01} - \frac{33}{2}(y_2 - 2) = 6.18;$

$2(y_2 - 2.97) - 0.33(y_2 - 2) = 0.1236;$

$1.67y_2 = 0.1236 + 5.28;$

$y_2 = 3.23569$ or 3.24 to 2dp