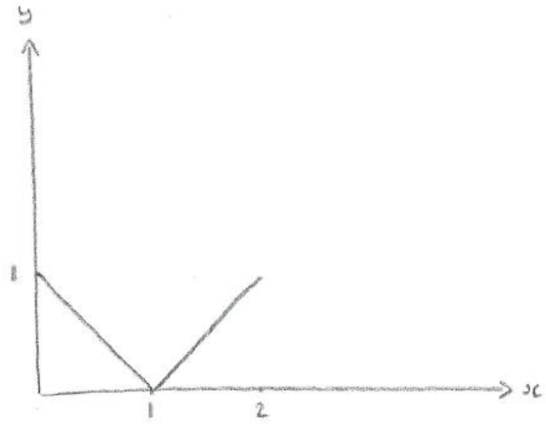
Curve Sketching - Exercises (Sol'ns)(11 pages; 10/4/20)

(1***) Sketch the graph of $\sqrt{x^2-2x+1}$ for $0 \leq x \leq 2$

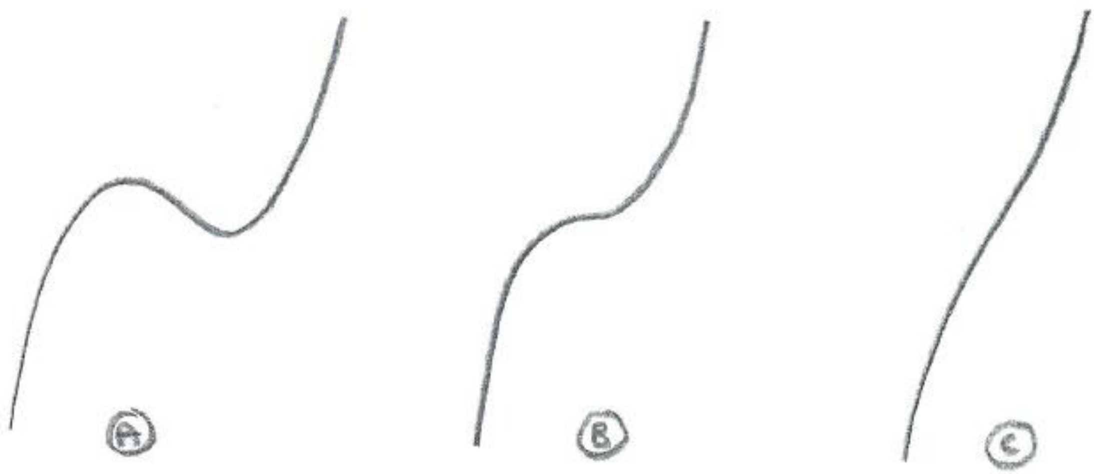
Solution: (see sketch below)

For $0 \leq x \leq 1, \sqrt{x^2-2x+1} = \sqrt{(x-1)^2} = \sqrt{(1-x)^2} = 1-x$

For $1 \leq x \leq 2, \sqrt{x^2-2x+1} = \sqrt{(x-1)^2} = x-1$



(2***) (i) What possible shapes might a cubic have (ignoring its position relative to the axes)?



A, B & C have 2, 1 & 0 stationary points respectively. These are for cases where the coefficient of x^3 is positive; so inverted shapes are also possible.

(ii) How many stationary points does the cubic function,

$f(x) = x^3 + x^2 - 2x + 3$ have?

$f'(x) = 3x^2 + 2x - 2$

To find the number of solutions to $f'(x) = 0$,

consider the discriminant: $2^2 - 4(3)(-2) > 0$

Thus there are 2 stationary points.

(iii) What is the condition for there to be 2 stationary points for the general cubic $f(x) = ax^3 + bx^2 + cx + d$?

2 sol'ns of $f'(x) = 3ax^2 + 2bx + c = 0$

$\Rightarrow (2b)^2 - 4(3a)c > 0$

$\Rightarrow b^2 - 3ac > 0$

(iv) For $f(x) = ax^3 + bx^2 + cx + d$, find the x-coordinate of any turning points of the gradient.

For a stationary point of the gradient, we want $\frac{d}{dx}(f'(x)) = 0$; ie $f''(x) = 0$:

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$f''(x) = 0 \Rightarrow x = -\frac{b}{3a}$

And $\frac{d^2}{dx^2}\left(f'(x)\right) = f'''(x) = 6a > 0$, so that the stationary point is a minimum (ie it is a turning point).

A turning point of the gradient is the definition of a point of inflexion (or inflection).

Thus, all cubics have one point of inflexion. They can be shown to have rotational symmetry about this point.

If the cubic has turning points, how could they be used to find the point of inflexion?

By symmetry, the coordinates of the point of inflexion will be halfway between those of the turning points.

(v) For $f(x) = ax^3 + bx^2 + cx + d$, find conditions for the shape of the curve to be each of the 3 possibilities shown in (i), by considering the gradient at the point of inflexion.

$f'\left(-\frac{b}{3a}\right) = \frac{b^2}{3a} - \frac{2b^2}{3a} + c = c - \frac{b^2}{3a}$

Diagram (A): Either (i) $a > 0$ & $f'\left(-\frac{b}{3a}\right) < 0$

or (ii) $a < 0$ & $f'\left(-\frac{b}{3a}\right) > 0$

(i): $3\text{ac} - b^2 < 0 \Leftrightarrow b^2 - 3ac > 0$ [agreeing with part (iii)]

(ii): $3\text{ac} - b^2 < 0$ also

Diagram (B): Stationary point of inflexion $\Leftrightarrow$ $f'\left(-\frac{b}{3a}\right) = 0$

$\Leftrightarrow b^2 - 3ac = 0$

Diagram (C): Either $(i)\ a > 0\ \&\ f'\left(-\frac{b}{3a}\right) > 0$

or (ii) $a < 0\ \&\ f'\left(-\frac{b}{3a}\right) < 0$,

so that $b^2 - 3ac < 0$

(3**) Sketch $y = |x-2| + 1$

Solution

Method 1

Case (i) $x - 2 \geq 0$

$y = |x-2| + 1 = (x-2) + 1 = x - 1$ for $x \geq 2$

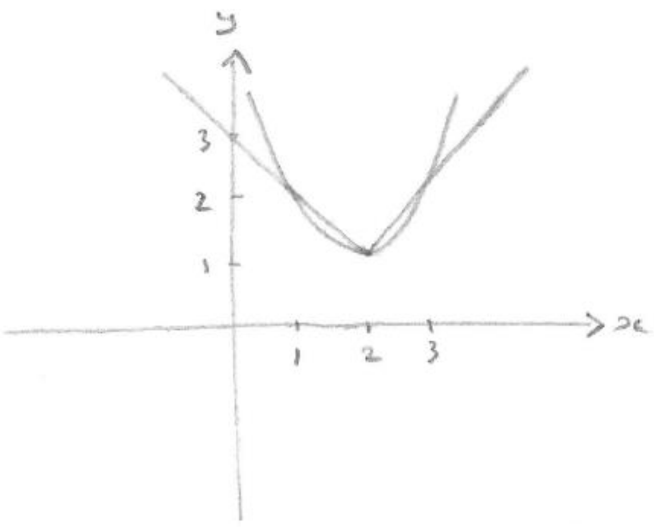
Case (ii) $x - 2 < 0$

$y = |x-2| + 1 = -(x-2) + 1 = 3 - x$ for $x < 2$

(The two lines will meet when $x = 2, y = 1$)

Method 2

Informally, $y = |x-2| + 1$ will behave similarly to $y = (x-2)^2 + 1$, and will have a minimum at (2,1)



(4***)(i) Sketch the curve $y = \frac{4x^2+5x+7}{2x+3}$.

(ii) Without using calculus, find the coordinates of the stationary points (to 3sf).

Solution

(i) Step 1: $x = 0 \Rightarrow y = \frac{7}{3}$;

$4x^2 + 5x + 7 = 4(x + \frac{5}{8})^2 - \frac{25}{16} + 7 > 0$, so that there are no intersections with the x-axis

Step 2: vertical asymptote when $2x + 3 = 0 \Rightarrow x = -\frac{3}{2}$

$x = -\frac{3}{2} + \delta$ ($\delta > 0$ is small) $\Rightarrow y = \frac{+}{+}$; ie $y > 0$

$x = -\frac{3}{2} - \delta \Rightarrow y = \frac{+}{-}$; ie $y < 0$

Step 3: To find $\lim_{x\to\infty} \frac{4x^2+5x+7}{2x+3}$:

$\frac{4x^2+5x+7}{2x+3} = \frac{4x^2+6x}{2x+3} + \frac{-x+7}{2x+3} = 2x + \frac{-x+\frac{3}{2}}{2x+3} + \frac{-\frac{3}{2}+7}{2x+3}$

$= 2x - \frac{1}{2} + \frac{11}{2(2x+3)}$ (*)

So $\lim_{x\to\infty} \frac{4x^2+5x+7}{2x+3} = 2x - \frac{1}{2}$

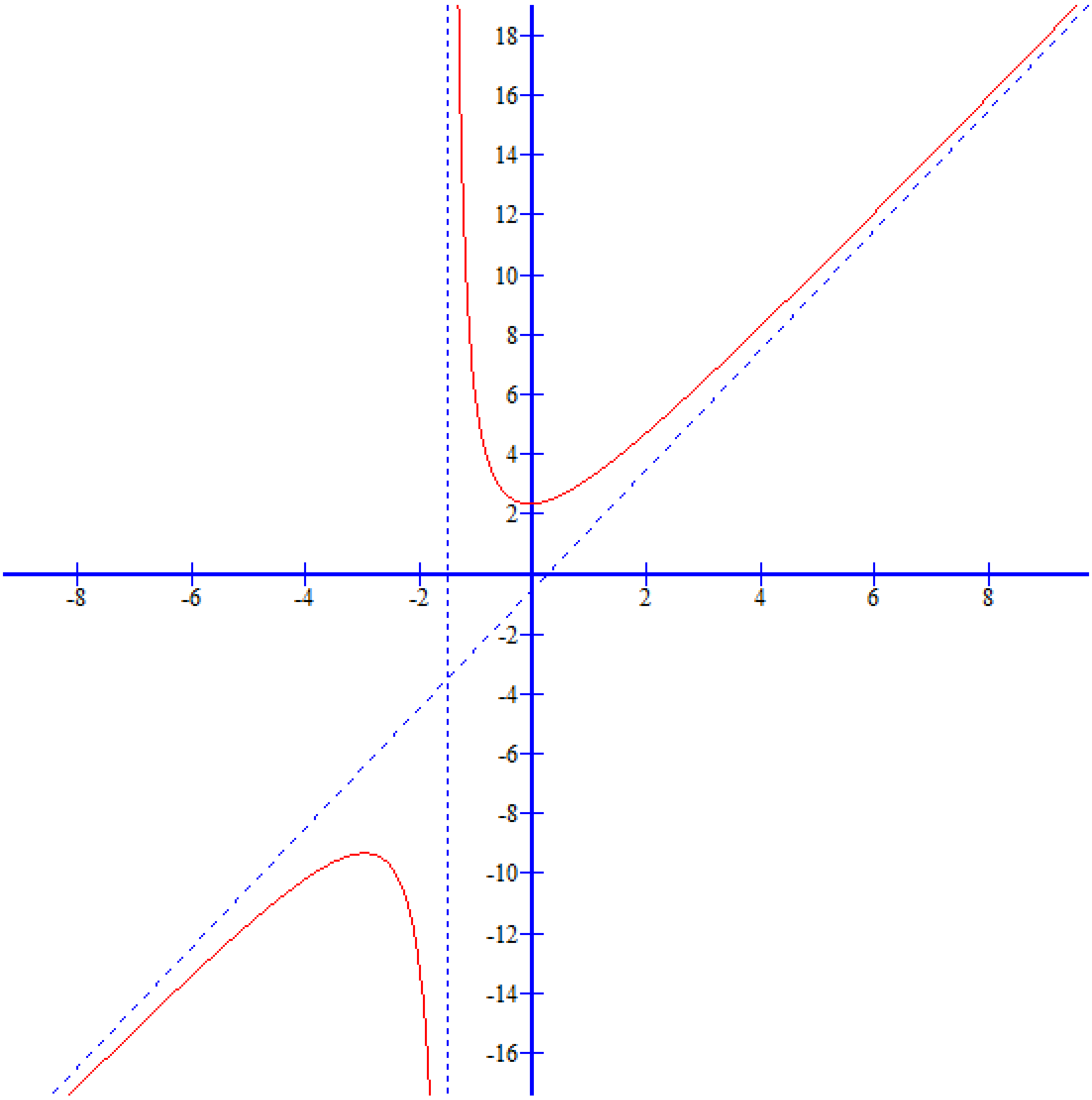
ie there is an oblique asymptote of $y = 2x - \frac{1}{2}$

(also approached as $x \to -\infty$)

and, from (*), as $x \to \infty, y > 2x - \frac{1}{2}$, and as $x \to -\infty, y < 2x - \frac{1}{2}$

[Note: We can't say $\lim_{x\to\infty} \frac{4x^2+5x+7}{2x+3} = \lim_{x\to\infty} \frac{4x+5+\frac{7}{x}}{2+\frac{3}{x}} = \frac{4x+5}{2} = 2x + \frac{5}{2}$

as $lim\frac{f(x)}{g(x)} = \frac{lim\, f(x)}{lim\, g(x)}$ only when $\lim f(x)$ and $\lim g(x)$ are constants.]



(ii) To find the stationary points, consider the values of x for which $\frac{4x^2+5x+7}{2x+3} = k$ has repeated roots;

Then $4x^2 + 5x + 7 = k(2x + 3)$

and $4x^2 + x(5 - 2k) + 7 - 3k = 0$,

with repeated roots occurring when the discriminant is zero, so that $(5-2k)^2-16(7-3k)=0$

$\Rightarrow 4k^2+k(-20+48)+25-112=0$

ie $4k^2+28k-87=0$

$\Rightarrow k=\frac{-28\pm\sqrt{28^2-4(4)(-87)}}{8}=2.33095$ or -9.33095

The corresponding x-coordinates are $\frac{-(5-2k)}{8}$;

ie -0.042263 and -2.95774

So there is a local minimum at $(-0.0423,2.33)$ and a local maximum at $(-2.96,-9.33)$ (3sf).

(5***)(i) Find a series of transformations that can be applied to $y=\frac{1}{x}$ to produce $y=\frac{3x-2}{6x-1}$.

(ii) Hence or otherwise, sketch the curve $y=\frac{3x-2}{6x-1}$.

Solution

(i) $\frac{3x-2}{6x-1}=\frac{3x-\frac{1}{2}-\frac{3}{2}}{6x-1}=\frac{1}{2}-\frac{3}{12}\left(\frac{1}{x-\frac{1}{6}}\right)$

So a possible series of transformations is:

a translation of $\begin{pmatrix}\frac{1}{6}\\0\end{pmatrix}$, followed by

a stretch of scale factor $\frac{1}{4}$ in the y-direction, followed by

a reflection in the x-axis, followed by

a translation of $\begin{pmatrix}0\\\frac{1}{2}\end{pmatrix}$

[Note: $\frac{1}{2}-\frac{3}{12}\left(\frac{1}{x-\frac{1}{6}}\right)=\frac{1}{2}-\frac{1}{4x-\frac{2}{3}}$, so an alternative series of transformations is:

a translation of $\begin{pmatrix}\frac{2}{3}\\0\end{pmatrix}$ $[\frac{1}{x}\rightarrow\frac{1}{x-\frac{2}{3}}]$ followed by

a stretch of scale factor $\frac{1}{4}$ in the x-direction $[\frac{1}{x-\frac{2}{3}}\rightarrow\frac{1}{4x-\frac{2}{3}}]$, followed by a reflection in the x-axis, followed by a translation of $\begin{pmatrix}0\\\frac{1}{2}\end{pmatrix}$.

Alternatively, $\frac{1}{4x-\frac{2}{3}}$ could be obtained instead by a stretch of scale factor $\frac{1}{4}$ in the x-direction $[\frac{1}{x}\rightarrow\frac{1}{4x}]$ (or a stretch of scale factor $\frac{1}{4}$ in the y-direction

$[\frac{1}{x}\rightarrow\frac{1}{4}(\frac{1}{x})]$), followed by a translation of $\begin{pmatrix}\frac{1}{6}\\0\end{pmatrix}$ $[\frac{1}{4x}\rightarrow\frac{1}{4(x-\frac{1}{6})}]$.]

(ii) As an alternative to performing the transformations in (i):

Step 1: $x=0\Rightarrow y=2$; $y=0\Rightarrow x=\frac{2}{3}$

Step 2: vertical asymptote when $6x-1=0\Rightarrow x=\frac{1}{6}$

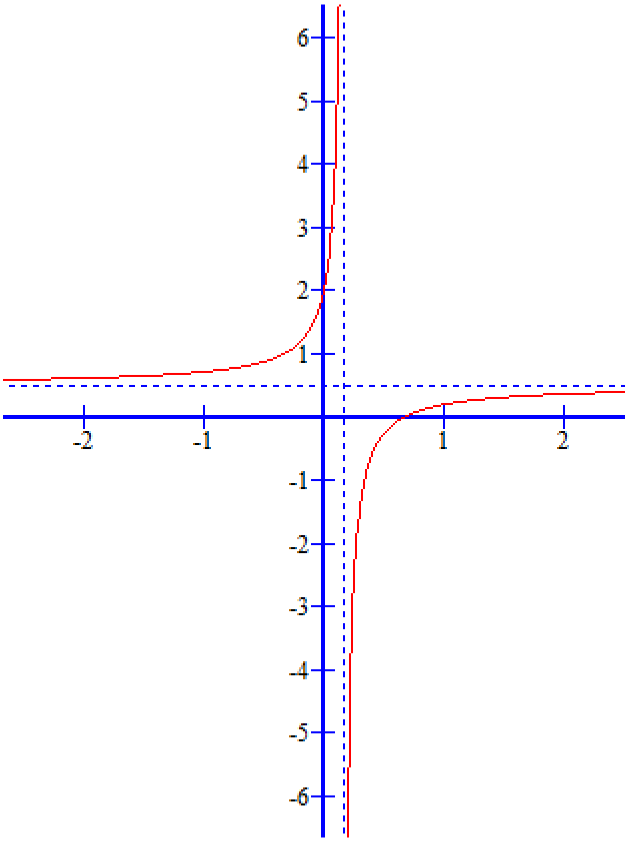
$x=\frac{1}{6}+\delta$ ($\delta>0$ is small) $\Rightarrow y=\frac{3x-2}{6x-1}=\frac{-}{+}$; ie $y<0$

$x=\frac{1}{6}-\delta\ \Rightarrow y=\frac{-}{-}$; ie $y>0$

Step 3: $\lim_{x\to\infty}\frac{3x-2}{6x-1}=\lim_{x\to\infty}\frac{3-\frac{2}{x}}{6-\frac{1}{x}}=\frac{3}{6}=\frac{1}{2}$ (and also as $x\to-\infty$)

Step 4: When $x=100, y=\frac{298}{599}<\frac{1}{2}$, so that $y\to\frac{1}{2}^{-}$ as $x\to\infty$

and when $x = -100, y = \frac{-302}{-601} > \frac{1}{2}$, so that $y \to \frac{1}{2}^+$ as $x \to -\infty$



(6***) Sketch the function $y = \frac{x^2}{x-1}$

Solution

The curve crosses the x-axis at $x = 0$ (twice), when $y = 0$.

There is a vertical asymptote at $x = 1$.

$x = 1 + \delta \Rightarrow y = \frac{+}{+} = +$

$x = 1 - \delta \Rightarrow y = \frac{+}{-} = -$

To determine the behaviour of the curve as $x \to \pm\infty$,

$y = \frac{x^2}{x-1} = \frac{x^2-1}{x-1} + \frac{1}{x-1} = x + 1 + \frac{1}{x-1}$

Thus, as $x \to \pm\infty$, $y \to x + 1$ (an 'oblique' asymptote).

[Note that we cannot say that $\lim_{x\to\infty} \frac{x^2}{x-1} = \lim_{x\to\infty} \frac{x}{1-\frac{1}{x}} = \frac{x}{1}$,

as $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}$, only when $\lim_{x\to\infty} f(x)$ =constant, and $\lim_{x\to\infty} g(x)$ =constant.]

To see how the curve approaches the oblique asymptote,

consider solutions of $\frac{x^2}{x-1} = x + 1 \Rightarrow x^2 = x^2 - 1$;

ie there are no points of intersection, and so the curve must be approaching the oblique asymptote from below as $x \to -\infty$, and from above as $x \to \infty$.

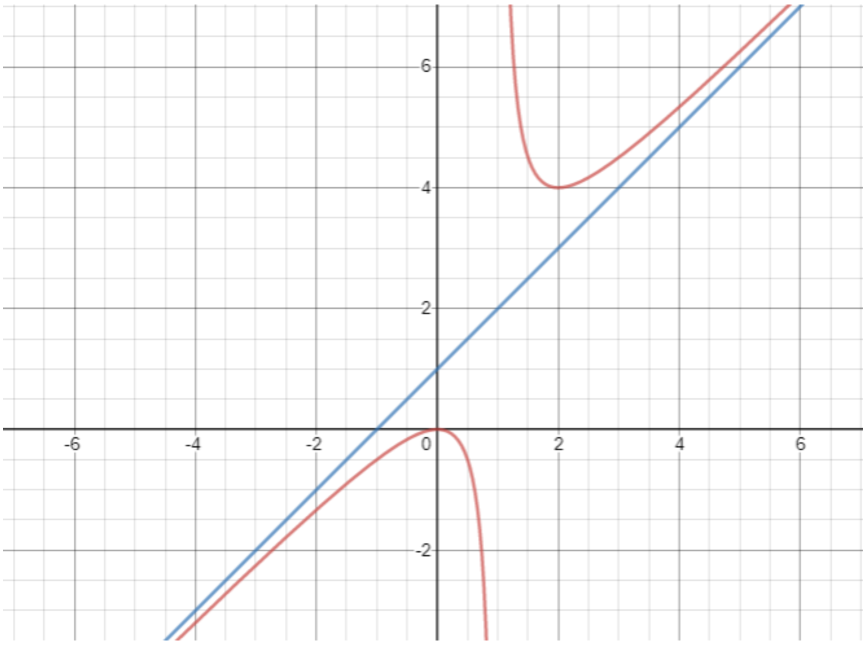
The local maximum of the curve will be at the Origin, where there is the repeated root of $y = 0$.

To find the location of the local minimum, consider solutions of $\frac{x^2}{x-1} = k$; ie $x^2 - kx + k = 0$

For there to be a solution, the discriminant must be non-negative; ie $(-k)^2 - 4k \geq 0 \Rightarrow k(k-4) \geq 0 \Rightarrow k \leq 0$ or $k \geq 4$

Thus there are no points of the curve for which $0 < y < 4$, and so the local minimum occurs when $y = 4$ (and $x^2 - 4x + 4 = 0$

$\Rightarrow x = 2$).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