Counting – Exercises (Sol'ns)

(5 pages; 7/7/20)

Key to difficulty:

* easier

** moderate

*** harder

(1**) An exam candidate is supposed to answer 4 questions out of 5 in Section A, 3 out of 4 in Section B, and 2 out of 3 in Section C. If they don't read the instructions properly and answer 9 questions at random from the paper, what is the probability that they answer the questions they are supposed to?

Solution

Number of ways of choosing 9 questions from the whole paper

$= \binom{12}{9} = \frac{12(11)(10)}{6} = 220$

Number of ways of choosing 4 questions out of 5 from Section A, 3 out of 4 from Section B, and 2 out of 3 from Section C

$= \binom{5}{4} \times \binom{4}{3} \times \binom{3}{2} = 5 \times 4 \times 3 = 60$

Prob (making the right choice)$= \frac{60}{220} = \frac{3}{11}$

(2***) (i) 3 different sweets are to be shared amongst 5 children. In how many ways can this be done, if no child is to receive all 3 sweets?

(ii) What is the answer if the 3 sweets are indistinguishable?

Solution

(i) **Method 1**

There are 5^3 ways of allocating the children to the 3 sweets. Then deduct the 5 ways in which one child has all 3 sweets, to give

$5^3 - 5 = 120$

Method 2

The permissible cases are: ABC, AAB, ABA, BAA (where AAB means that sweets 1 & 2 go to child A & sweet 3 goes to child B).

The total number of possibilities is then:

$5(4)(3) + 5(4) + 5(4) + 5(4)$ [for AAB eg, there are 5 ways of choosing A, and then 4 ways of choosing B]

$= 60 + 20 + 20 + 20 = 120$

(ii) If the 3 sweets are indistinguishable, there are 2 permissible cases:

AAB and ABC (where AAB, for example, means that child A receives 2 sweets & child B receives 1 sweet)

The total number of such cases is 5×4 (5 ways of choosing A, and then 4 ways of choosing B) + $\binom{5}{3} = 20 + 10 = 30$.

(3***) When choosing the venue for an international conference, 3 countries are shortlisted at random from a list of 9, of which 4 are European and 5 are from the rest of the world. What is the probability that at least 2 of the countries shortlisted are European?

Solution

Method 1a

Number of ways of selecting 3 countries out of 9

$= \binom{9}{3} = \frac{9(8)(7)}{6} = 3(4)(7) = 84$

Number of ways of selecting 2 European countries out of 4, and 1 non-European country out of 5 $= \binom{4}{2}\binom{5}{1} = 6(5) = 30$

Number of ways of selecting 3 European countries out of 4, and 0 non-European country out of 5 $= \binom{4}{3}\binom{5}{0} = 4(1) = 4$

Prob(at least 2 European countries) $= \frac{30+4}{84} = \frac{17}{42}$

Method 1b

Number of ways of selecting 1 European country out of 4, and 2 non-European countries out of 5 $= \binom{4}{1}\binom{5}{2} = 4(10) = 40$

Number of ways of selecting 0 European countries out of 4, and 3 non-European countries out of 5 $= \binom{4}{0}\binom{5}{3} = 1(10) = 10$

Prob(at least 2 European countries) $= 1 - \frac{40+10}{84} = 1 - \frac{25}{42} = \frac{17}{42}$

Method 2a

Prob(2 European countries)$=$ $3 \times$ Prob(EER) $= 3 \times \frac{4}{9} \times \frac{3}{8} \times \frac{5}{7} = \frac{5}{14}$

(where EER means that the 1^{st} & 2^{nd} countries selected are European, and the 3^{rd} is from the rest of the world)

Prob(3 European countries)$= \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7} = \frac{1}{21}$

Hence Prob(at least 2 European countries) $= \frac{5}{14} + \frac{1}{21} = \frac{15+2}{42} = \frac{17}{42}$

Method 2b

Prob(0 European countries) $= \frac{5}{9} \times \frac{4}{8} \times \frac{3}{7} = \frac{5}{3\times2\times7} = \frac{5}{42}$

Prob(1 European country)

$= 3 \times \text{Prob(ERR)} \quad = 3 \times \frac{4}{9} \times \frac{5}{8} \times \frac{4}{7} = \frac{5\times2}{3\times7} = \frac{10}{21}$

Hence Prob(at least 2 European countries) $= 1 - \frac{5}{42} - \frac{10}{21} = 1 - \frac{25}{42} = \frac{17}{42}$

(4***) The following books are on a bookshelf: 4 novels, 3 history books, 2 biographies and 1 dictionary. In how many ways can they be arranged if the novels have to be together, and similarly for the history books and biographies?

Solution

[Note that we treat the novels etc as being distinguishable from each other.]

There are 4! ways of arranging the items N, H, B & D. Then to allow for the 4! ways of arranging the novels etc, we multiply by 4!3!2!, to give: 4!4!3!2! = 6912

(5**) 3 bananas, 4 apples and 5 oranges are to be arranged in a row. In how many ways can this be done, assuming that the bananas etc are indistinguishable?

Solution

There are 12! ways of ordering the items, where the bananas etc **are** distinguishable. Divide by 3! to remove the duplication, as far as the bananas are concerned, and similarly for the others, to give:

$\frac{12!}{3!4!5!} = \frac{12(11)(10)(9)(8)(7)(6)}{6(24)} = (11)(5)(9)(8)(7) = 27720$