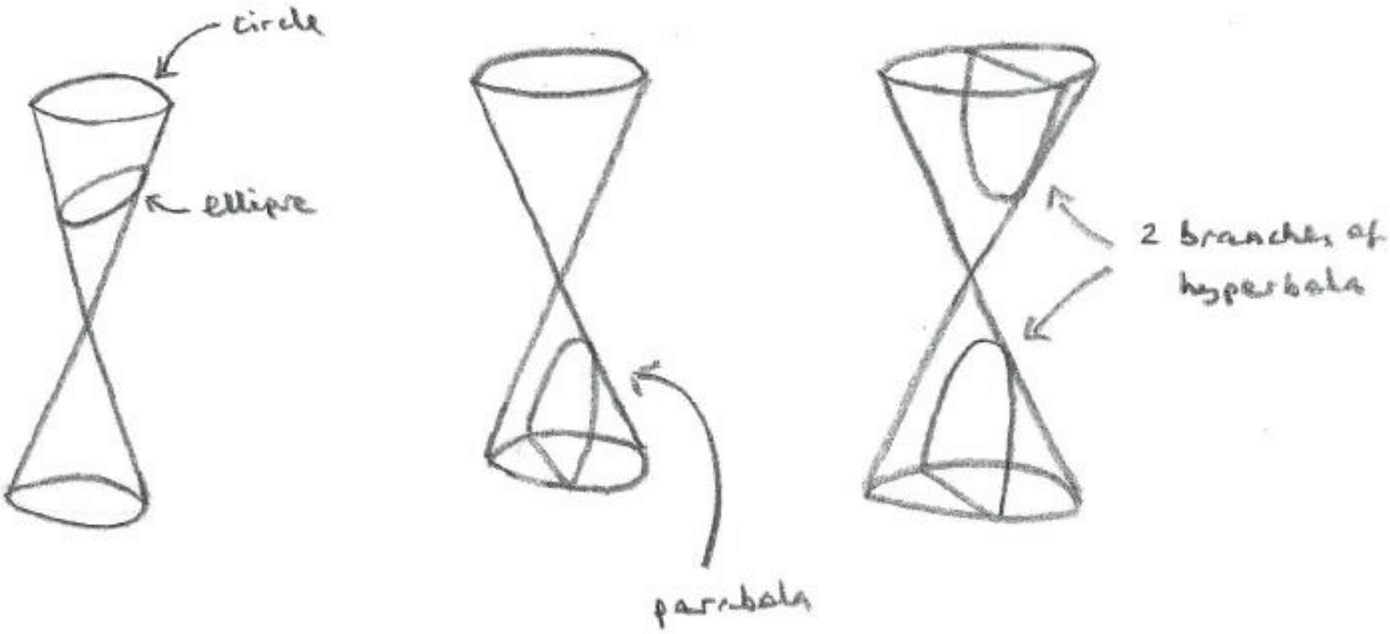
Conics (4 pages; 17/8/19)

This note relates to the general conic and is concerned with properties common to parabolas, ellipses and hyperbolas (as well as circles). See also the separate notes "Parabolas", "Ellipses" and "Hyperbolas".

(1) Conic Sections

Circles, ellipses, parabolas and hyperbolas can be interpreted as the intersection of a plane with a cone (together with an inverted cone) - or 'conic section'. Thus these four curves are sometimes referred to as 'conics'.



Thus the circle occurs where the plane is horizontal; the ellipse occurs where the plane is tilted, but such that the angle of tilt isn't as great as the angle made by the cone. In the case of the parabola, the angle of tilt is the same as that of the cone; whilst in the case of the hyperbola, the angle of tilt is greater than that of the cone (so that the plane cuts through the inverted cone as well, to give the two branches of the hyperbola.)

(2) Eccentricity of conic

Let P be a point on a conic. Let S be a point, which we will call a '**focus**' of the conic. Let PM be the perpendicular distance from P to a line, which we will call a '**directrix**' of the conic. A general conic can be defined as the **locus** of points such that $\mathrm{e} = \frac{PS}{PM}$ (called the eccentricity) is constant.

If $\mathrm{e} = 0$, the curve is a circle.

If $0<\mathrm{e}<1$, the curve is an ellipse.

If $\mathrm{e}=1$, the curve is a parabola.

If $\mathrm{e}>1$, the curve is a hyperbola.

($e = \infty$ gives a straight line)

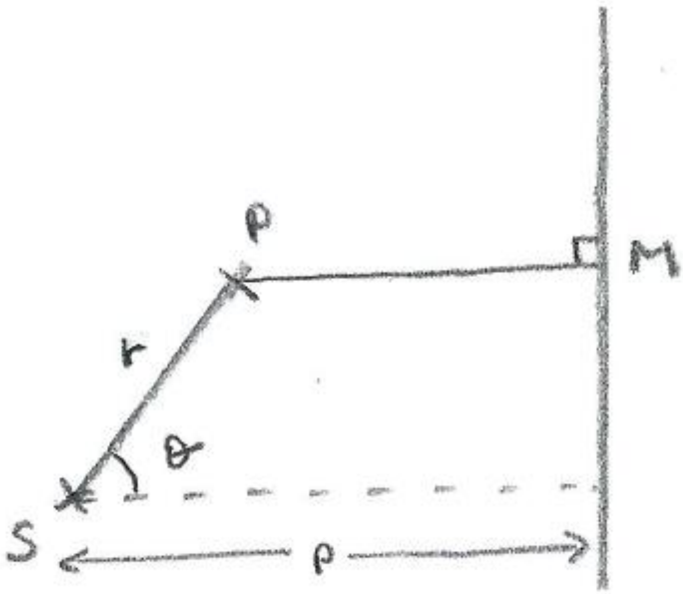
It turns out that there is only one focus and one directrix for a parabola (whereas there are two of both for an ellipse or hyperbola).

(3) Polar form of general conic

If the pole is taken to be a focus of the conic, then

$\mathrm{e} = \frac{PS}{PM} \Rightarrow e = \frac{r}{p - r\cos\theta}$ (referring to the diagram below),

where p is the distance from the focus to the directrix



so that $ep - ercos\theta = r$ and $r(1 + ecos\theta) = ep$,

giving $r = \frac{ep}{1+ecos\theta}$

This formula is applicable in the case that the directrix is vertical and lies to the right of the focus.

(4) Variations of the polar form

If the directrix is horizontally above the pole, then the equation can be obtained as followed:

θ is measured from the positive x-axis. We want r to have the same value at $\theta = \frac{\pi}{2}$ as it had at $\theta = 0$ for $r = \frac{ep}{1+ecos\theta}$ (since the hyperbola has just been rotated by $\frac{\pi}{2}$), and generally lagging behind $r = \frac{ep}{1+ecos\theta}$ by an angle of $\frac{\pi}{2}$. As $sin\theta$ lags behind $cos\theta$ by an angle of $\frac{\pi}{2}$, the required equation is $r = \frac{ep}{1+esin\theta}$

Alternatively, we could have written

$$r = \frac{ep}{1+ecos(\theta-\frac{\pi}{2})} = \frac{ep}{1+ecos(\frac{\pi}{2}-\theta)} = \frac{ep}{1+esin\theta}$$

(or just by the compound angle formula)

Similarly, if there is a vertical directrix to the left of the pole, we can think of r at angle θ being r at angle$(\theta - \pi)$ in the original equation, so that the required equation is

$$r = \frac{ep}{1+ecos(\theta-\pi)} = \frac{ep}{1+ecos(\pi-\theta)} = \frac{ep}{1-ecos\theta}$$

And finally, if there is a horizontal directrix below the pole,

$r = \frac{ep}{1+ecos(\theta+\frac{\pi}{2})} = \frac{ep}{1-esin\theta}$ (eg by the compound angle formula)

(5) Orbits

The orbits of planets and returning comets (such as Halley's comet) about the Sun are ellipses, with one focus being the Sun.

For the Earth, $e \approx 0.02$ (it varies a bit over long periods of time), so that the orbit is virtually circular.

The path of a non-returning comet is either one branch of a hyperbola, or a parabola. In both cases the Sun is at the appropriate focus.