Complex Numbers - Part 3 (12 pages; 1/5/25)

(20) De Moivre's Theorem

The theorem states that, if $z = cos\theta + isin\theta$, then

$z^n = cos(n\theta) + isin(n\theta)$, where n can be fractional and/or negative

When n is a positive integer, this follows from the result established earlier that, where $z_1 = r_1(cos\theta + isin\theta)$ and

$z_2 = r_2(cos\phi + isin\phi)$, then

$z_1z_2 = r_1r_2\{cos(\theta + \phi) + isin(\theta + \phi)\}$

Putting $z = z_1 = z_2$ gives $z^2 = cos(2\theta) + isin(2\theta)$, and this can be extended to higher integers by the same method.

Exercise: Express $(1 - i)^6$ in the form $x + iy$

Solution

First of all, express $z = 1 - i$ in modulus-argument form:

By considering the Argand diagram, $|z| = \sqrt{2}$ & $\arg(z) = -\frac{\pi}{4}$

So $z = \sqrt{2}(\cos\left(-\frac{\pi}{4}\right) + isin\left(-\frac{\pi}{4}\right))$

Then, by de Moivre's theorem,

$z^6 = \left(\sqrt{2}\right)^6 \left(\cos\left(-\frac{6\pi}{4}\right) + isin(-\frac{6\pi}{4})\right)$

$= 8\left(\cos\left(-\frac{3\pi}{2}\right) + isin\left(-\frac{3\pi}{2}\right)\right)$

$= 8\left(\cos\left(\frac{\pi}{2}\right) + isin\left(\frac{\pi}{2}\right)\right) = 8i$

When n is a negative integer:

Let $n = -k$

Then $(cos\theta + isin\theta)^n = \frac{1}{(cos\theta+isin\theta)^k} = \frac{1}{cosk\theta+isink\theta}$

$= \frac{1}{cosk\theta+isink\theta} \cdot \frac{cosk\theta-isink\theta}{cosk\theta-isink\theta} = \frac{\cos(-k\theta)+isin(-k\theta)}{cos^2k\theta+sin^2k\theta}$

$= \cos(n\theta) + isin(n\theta)$

Results following from de Moivre's theorem

(i) $(cos\theta - isin\theta)^n = (\cos(-\theta) + isin(-\theta))^n$

$= \cos(-n\theta) + isin(-n\theta) = \cos(n\theta) - isin(n\theta)$

(ii) If $z = cos\theta + isin\theta$,

then $z^{-1} = \cos(-\theta) + isin(-\theta) = cos\theta - isin\theta = z^*$

(but note that $z^{-1} = z^*$ only when $|z| = 1$; $zz^* = |z|^2$ also gives this result)

(iii) For general $z = r(cos\theta + isin\theta)$, $z^{-1} = \frac{1}{r}(cos\theta - isin\theta)$

$= \frac{1}{r} \cdot \frac{z^*}{r} = \frac{z^*}{|z|^2}$

De Moivre's theorem can also be shown to be true for fractional n.

(21) Using de Moivre's Theorem to establish Trig. identities: Multiple angle formulae

Example: Show that $cos2\theta = cos^2\theta - sin^2\theta$

$cos2\theta = Re\{cos2\theta + isin2\theta\} = Re\{(cos\theta + isin\theta)^2\}$

$= Re\{cos^2\theta + 2icos\theta sin\theta - sin^2\theta\}$

$= cos^2\theta - sin^2\theta$

(and similarly $sin2\theta = 2sin\theta cos\theta$)

Exercise: Find an expression for $sin3\theta$ in terms of powers of $sin\theta$ and/or $cos\theta$

Solution

$sin3\theta = Im(cos3\theta + isin3\theta)$

$cos3\theta + isin3\theta = (cos\theta + isin\theta)^3$

$= cos^3\theta + 3cos^2\theta(isin\theta) + 3cos\theta(isin\theta)^2 + (isin\theta)^3$

Hence $sin3\theta = 3cos^2\theta(sin\theta) - sin^3\theta$

$= 3(1 - sin^2\theta)(sin\theta) - sin^3\theta$

$= 3sin\theta - 4sin^3\theta$

(22) Powers of Sines and Cosines

Powers of Cosines

To find $cos^2\theta$ in terms of $cos2\theta$:

Starting point: $cos\theta = \frac{1}{2}(z + z^{-1})$,

where z $= cos\theta + isin\theta$ and $z^{-1} = cos\theta - isin\theta$

Then $cos^2\theta = \frac{1}{4}(z + z^{-1})^2 = \frac{1}{4}(z^2 + 2 + z^{-2})$

Now $z^2 + z^{-2} = (cos2\theta + isin2\theta) + (cos2\theta - isin2\theta) = 2cos2\theta$

Hence $cos^2\theta = \frac{1}{4}(2 + 2cos2\theta) = \frac{1}{2}(1 + cos2\theta)$

Exercise: Show that $cos^3\theta = \frac{1}{4}(cos3\theta + 3cos\theta)$

Solution

$cos\theta = \frac{1}{2}(z + z^{-1})$

where z $= cos\theta + isin\theta$ and $z^{-1} = cos\theta - isin\theta$

So $cos^3\theta = \frac{1}{8}(z + z^{-1})^3 = \frac{1}{8}\ (z^3 + 3z + 3z^{-1} + z^{-3})$

$= \frac{1}{8}\{3(z + z^{-1}) + (z^3 + z^{-3})\}$

$= \frac{1}{8}\{3(2cos\theta) + (2cos3\theta)\}$

$= \frac{1}{4}(cos3\theta + 3cos\theta)$

Powers of Sines

$isin\theta = \frac{1}{2}(z - z^{-1}),$

where z $= cos\theta + isin\theta$ and $z^{-1} = cos\theta - isin\theta$

So $-isin^3\theta = \frac{1}{8}(z - z^{-1})^3$ (1)

Exercise: Find an expression for $sin^3\theta$

Solution

$(1) \Rightarrow -8isin^3\theta = z^3 - 3z + 3z^{-1} - z^{-3}$

$= z^3 - z^{-3} - 3(z - z^{-1})$

$= 2isin(3\theta) - 3(2i)sin\theta$

Hence $sin^3\theta = -\frac{1}{8}\ (2\sin(3\theta) - 6sin\theta) = \frac{1}{4}(3sin\theta - \sin(3\theta))$

(23) Exponential form of complex number

$z = r(cos\theta + isin\theta) = re^{i\theta}$

Justification (assuming knowledge of Maclaurin expansions of $e^x, cos\ x$ & $\sin x$):

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} \ldots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots)$$

$$= cos\theta + isin\theta$$

De Moivre's theorem is then simply: $\left(e^{i\theta}\right)^n = e^{i(n\theta)}$, as we would expect.

(24) Roots of Complex Numbers

Consider the equation $z^3 = cos\theta + isin\theta$

Then $z = cos\left(\frac{\theta}{3}\right) + isin\left(\frac{\theta}{3}\right)$ is a solution

But $cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) + isin\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)$ is a solution as well

and so is $cos\left(\frac{\theta}{3} + 2(\frac{2\pi}{3})\right) + isin\left(\frac{\theta}{3} + 2(\frac{2\pi}{3})\right)$

These are the solutions of $z = (cos\theta + isin\theta)^{1/3}$

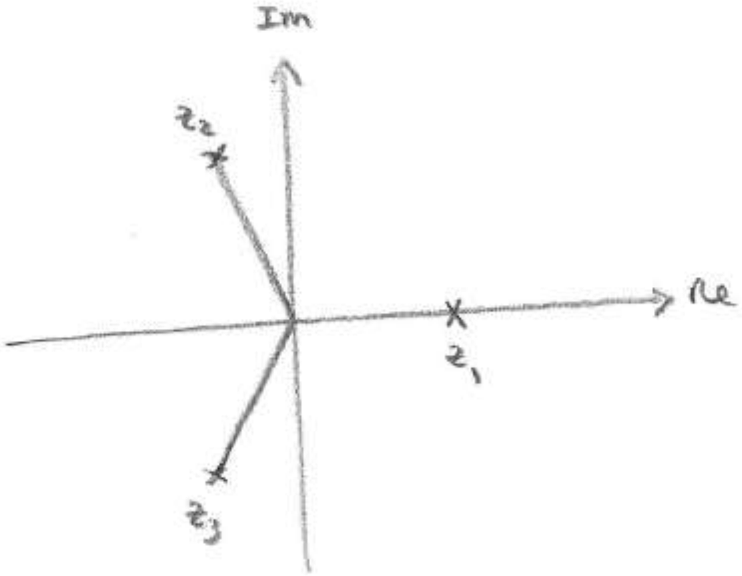
Exercise: When $\theta = 0$, express these 3 solutions in the form $a + bi$, and show them on the Argand diagram.

Solution

$$z_1 = cos\left(\frac{0}{3}\right) + isin\left(\frac{0}{3}\right) = 1$$

$$z_2 = cos\left(\frac{0}{3} + \frac{2\pi}{3}\right) + isin\left(\frac{0}{3} + \frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$z_3 = cos\left(\frac{0}{3} + 2(\frac{2\pi}{3})\right) + isin\left(\frac{0}{3} + 2(\frac{2\pi}{3})\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$



So there are 3 solutions of $z^3 = cos\theta + isin\theta$, spread evenly on a unit circle in the Argand diagram, starting at $\frac{\theta}{3}$. These are the 3 cube roots of $cos\theta + isin\theta$.

More generally, there will be n roots of the equation

$z^n = r(cos\theta + isin\theta)$;

namely $z = r^{\frac{1}{n}}(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + isin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right))$

for $k = 0,1, \dots, n-1$

Note that $\frac{\theta}{n} + \frac{2n\pi}{n} = \frac{\theta}{n} + 2\pi$, and so the root associated with $k = n$ is identical to that associated with $k = 0$

(25) When r is a non-negative real number, $\sqrt{r}$ is defined to be the positive square root (so that the solutions of $x^2 = r$ are $x = \pm\sqrt{r}$).

Because complex numbers are represented by points in the Argand diagram (in contrast to real numbers, which are represented by points on a number line), multiplication by -1 has a more complicated interpretation; namely as a rotation of 180°.

The square root of the complex number $z = re^{i\theta}$

$(r \geq 0 \;\&\; -\pi < \theta \leq \pi)$ is defined as $\sqrt{z} = \sqrt{r}e^{i\theta/2}$, and the solutions of $u^2 = z$ are $u = \pm\sqrt{r}e^{i\theta/2}$.

However, the complex square root function is not continuous, as when $z = e^{i\pi}, \sqrt{z} = e^{i\pi/2} = i$, whilst for the neighbouring point in the Argand diagram, $z = e^{-i(\pi-\delta)}, \sqrt{z} = e^{-i(\pi-\delta)/2}$, which is close to $e^{-i\pi/2} = -i$. It can be shown that, for this reason, it is not generally true that $\sqrt{uv} \neq \sqrt{u}\sqrt{v}$. For example, $\sqrt{-1}\sqrt{-1} = i^2 = -1$, but $\sqrt{(-1)(-1)} = \sqrt{1} = 1$.

(26) Relation between the roots of unity

Example: The 5 roots of $z^5 = 1$ (the "roots of unity") are

$cos\theta + isin\theta$, where $\theta = \frac{2k\pi}{5}$, for $k = 0,1, \dots ,4$

The 1st root after 1 is commonly denoted by ω,

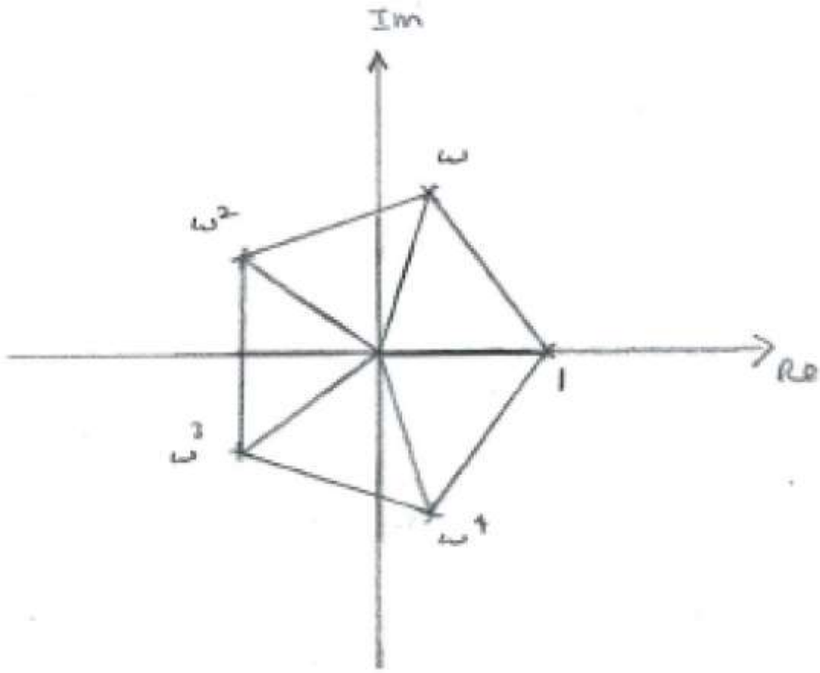
so that $\omega = cos\left(\frac{2\pi}{5}\right) + isin\left(\frac{2\pi}{5}\right)$

Then $\omega^2 = cos\left(\frac{4\pi}{5}\right) + isin\left(\frac{4\pi}{5}\right)$, by de Moivre's theorem.

In general, $\omega^k = cos\left(\frac{2k\pi}{5}\right) + isin\left(\frac{2k\pi}{5}\right)$,

and we can see that the 5 roots are: $1, \omega, \omega^2, \omega^3 \;\&\; \omega^4$

These form the vertices of a polygon, as in the diagram below.



The following result will now be proved:

$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

Approach 1 (Polynomial roots & coefficients)

The polynomial equation being solved is $z^5 - 1 = 0$, and the sum of the roots is $' - \frac{b'}{a} = 0$

Approach 2 (algebraic)

This is a geometric series with common ratio ω, and so

$LHS = \frac{\omega^5 - 1}{\omega - 1} = \frac{0}{\omega - 1}$ (as $\omega^5 = 1$) $= 0$ (as $\omega \neq 1$)

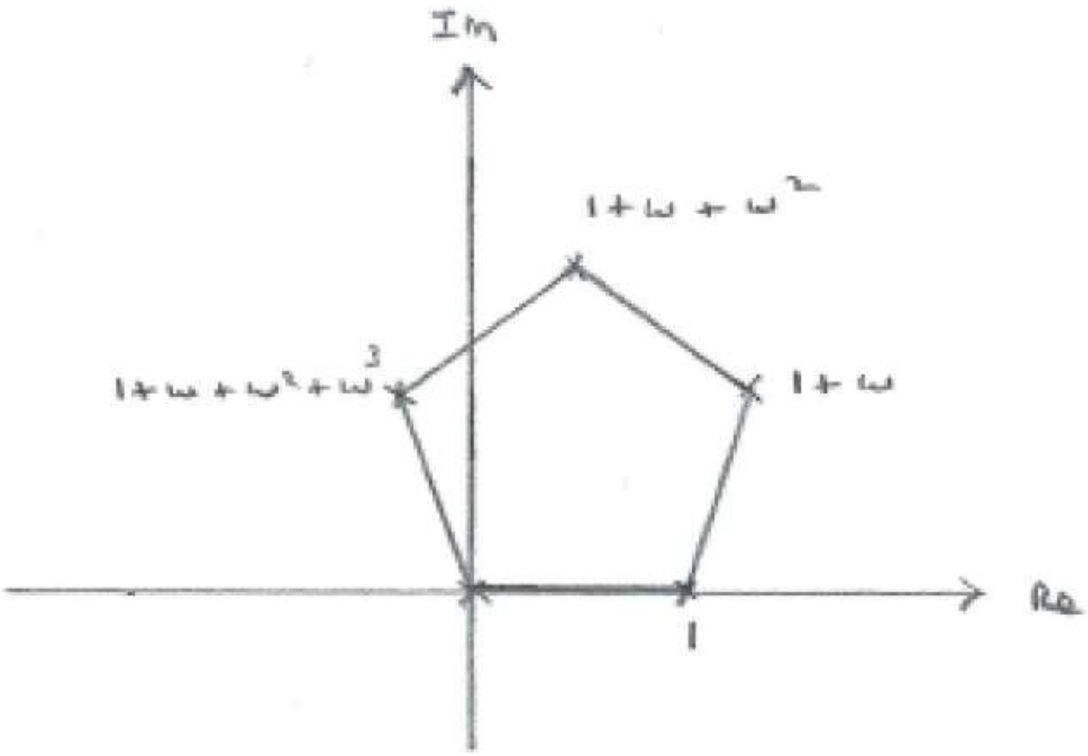
Approach 3 (vectorial)

Treating complex numbers as vectors, $1+\omega$ can be created as a vertex of the (new) polygon shown below. This then leads to

$1+\omega+\omega^2$, and so on.

The 5 sides of the polygon are $1, \omega, \omega^2, \omega^3$ & ω^4, in their vector form (each side has length 1, and the directions they make with the positive real axis are $0, \frac{2\pi}{5}, 2\left(\frac{2\pi}{5}\right), 3\left(\frac{2\pi}{5}\right), \ldots$)

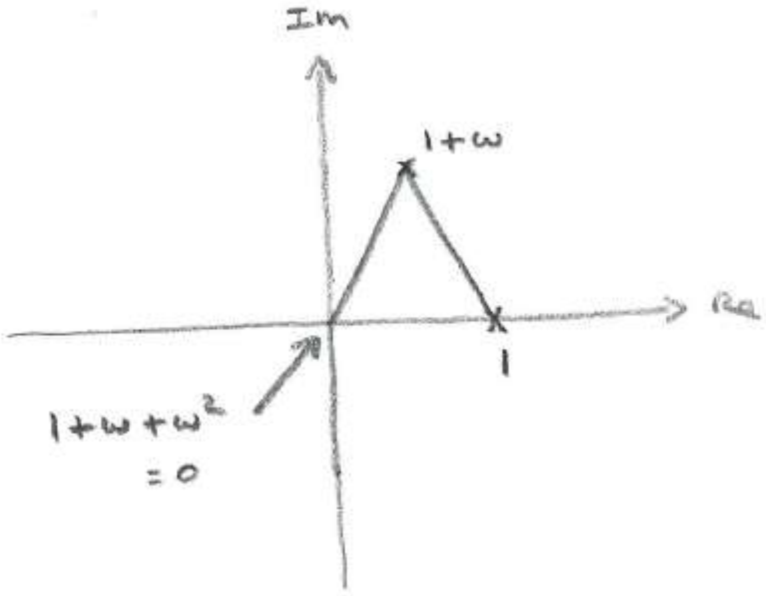
[Note that $1, \omega, \omega^2, \omega^3$ & ω^4 were the **vertices** of the 1st polygon.]



From the diagram we see that the vector $1+\omega+\omega^2+\omega^3+\omega^4$ is at the Origin; ie $1+\omega+\omega^2+\omega^3+\omega^4=0$

Exercise: If $1, \omega, \omega^2$ are the cube roots of 1, draw the polygon with vertices $1,\ 1+\omega,\ 1+\omega+\omega^2$

Solution



(27) Transformations from z-plane to w-plane

(i) Concerned with effect on loci in the z-plane.

(ii) $w = z + a + bi$: translation

(iii) $w = kz$: enlargement of scale factor $k(> 0)$ (centre the Origin)

(iv) Example (from Edx FP2, Ex 3H, Q4 - p59)

$w = 2z - 5 + 3i$; effect on the locus $|z - 2| = 4$?

($|z - 2| = 4$ can be written $(x - 2)^2 + y^2 = 16$)

Approach 1: enlargement of scale factor 2, followed by translation $-5 + 3i \Rightarrow$ centre of circle changes to 4, and then to $4 - 5 + 3i = -1 + 3i$; radius changes to 8 (translation has no effect)

Approach 2: $w = 2z - 5 + 3i \Rightarrow z = \frac{1}{2}(w + 5 - 3i)$

Then $|z-2|=4 \rightarrow \left|\frac{1}{2}(w+5-3i)-2\right|=4$

$\Rightarrow |w+1-3i|=8$ [or $(u+1)^2+(v-3)^2=64$]

(v) Example (from Edx FP2, Ex 3H, Q5(b))

$w=z-1+2i$; effect on locus $\arg(z-1+i)=\frac{\pi}{4}$?

Approach 1: All points on the half line are translated by $-1+2i$, with the direction of the line unchanged.

Approach 2: $w=z-1+2i \Rightarrow z=w+1-2i$

Then $\arg(z-1+i)=\frac{\pi}{4} \Rightarrow \arg(w+1-2i-1+i)=\frac{\pi}{4}$

$\Rightarrow \arg(w-i)=\frac{\pi}{4}$

(vi) Example (from Edx FP2, Ex 3H, Q5(c))

$w=z-1+2i$; effect on locus $y=2x$

Approach 1: as above

Approach 2: Consider separately $z=0, argz=tan^{-1}2$ &

$argz=(tan^{-1}2)-\pi$; then replace z with $w+1-2i$, as in (5).

(When $z=0, w=0-1+2i$)

Equation of line in w-plane is $\frac{y-2}{x-(-1)}=2$, as line passes through $-1+2i$, with the same gradient as before.

(vii) Example (from Edx FP2, Ex 3H, Q6(a))

$w=\frac{1}{z}$; effect on locus $|z|=2$?

$w=\frac{1}{z} \Rightarrow z=\frac{1}{w}$; then $|z|=2 \Rightarrow \left|\frac{1}{w}\right|=2 \Rightarrow |w|=\frac{1}{2}$

(viii) Example (from Edx FP2, Ex 3H, Q7(a))

$w = z^2$; show that going once round the circle $|z| = 3 \rightarrow$ going twice round the circle $|w| = 9$

$z = 3e^{\theta i}$ $(0 \leq \theta < 2\pi) \rightarrow w = 9e^{2\theta i}$ $(0 \leq 2\theta < 4\pi)$

(ix) Example (from Edx FP2, Ex 3H, Q12(a))

$w = \frac{-iz+i}{z+1}$; effect on $|z| = 1$?

$w = \frac{-iz+i}{z+1} \Rightarrow (z+1)w = -iz + i \Rightarrow z(w+i) = i - w$

$\Rightarrow z = \frac{i-w}{w+i}$

Then $|z| = 1 \Rightarrow |i - w| = |w + i|$; ie $|w - i| = |w + i|$