Complex Numbers Exercises (16 pages; 17/2/20)

(1*) Find $(2+5i) \div (1+3i)$ by two methods

Solution

Method 1

$\frac{2+5i}{1+3i} = \frac{(2+5i)(1-3i)}{(1+3i)(1-3i)} = \frac{2+15-6i+5i}{1+9} = \frac{17}{10} - \frac{i}{10}$

Check: $\frac{1}{10}(17-i)(1+3i) = \frac{1}{10}(17+3-i+51i) = 2+5i$

Method 2

Let $(2+5i) \div (1+3i) = a+bi$

Then $2+5i = (a+bi)(1+3i) = a+3ai+bi-3b$

Equating real parts: $2 = a - 3b$ (1)

Equating imaginary parts: $5 = 3a + b$ (2)

$(1) + 3 \times (2) \Rightarrow 17 = 10a \Rightarrow a = \frac{17}{10}$

Then $(2) \Rightarrow b = 5 - \frac{51}{10} = -\frac{1}{10}$

So $(2+5i) \div (1+3i) = \frac{17}{10} - \frac{i}{10}$

(2*) Solve the equation $(2+i)z + 3 = 0$ by two methods

Solution

Method 1

$(2+i)z + 3 = 0 \Rightarrow z = \frac{-3}{2+i} = \frac{-3(2-i)}{(2+i)(2-i)} = \frac{-6+3i}{4+1} = -\frac{6}{5} + \frac{3i}{5}$

Method 2

Let $z = a + bi$

Then $(2 + i)(a + bi) + 3 = 0$

$\Rightarrow 2a - b + (a + 2b)i + 3 = 0$

Equating real parts: $2a - b = -3$ (1)

Equating imaginary parts: $a + 2b = 0$ (2)

Substituting for a from (2) into (1), $2(-2b) - b = -3$ and $\therefore b = \frac{3}{5}$ and $a = -\frac{6}{5}$

so that $z = -\frac{6}{5} + \frac{3i}{5}$

(3*) Solve the equation $z^2 - 2z + 2 = 0$

(a) by completing the square
(b) by equating real & imaginary parts

Solution

(a) $z^2 - 2z + 2 = 0$

$\Rightarrow (z - 1)^2 + 1^2 = 0$

$\Rightarrow ([z - 1] + i)([z - 1] - i) = 0$

$\Rightarrow z = 1 - i \ or \ 1 + i$

(b) Let $z = a + bi$

Then $(a + bi)^2 - 2(a + bi) + 2 = 0$

$\Rightarrow a^2 - b^2 + 2abi - 2a - 2bi + 2 = 0$

equating real parts: $a^2 - b^2 - 2a + 2 = 0$ (1)

equating imaginary parts: $2ab - 2b = 0$ (2)

(2) $\Rightarrow b(a-1) = 0 \Rightarrow b = 0$ *or* $a = 1$

From (1), $b = 0 \Rightarrow a^2 - 2a + 2 = 0$

(this can be excluded, as a is real and there are no real solutions to the quadratic equation)

$a = 1 \Rightarrow 1 - b^2 = 0 \Rightarrow b = \pm 1$

Hence $z = 1 \pm i$

(4*) Represent the following on the Argand diagram:

(i) $|z - i| > |z + 1|$

(ii) $|z - i| = 2|z + 1|$

Solution

(i) Let $z = x + yi$

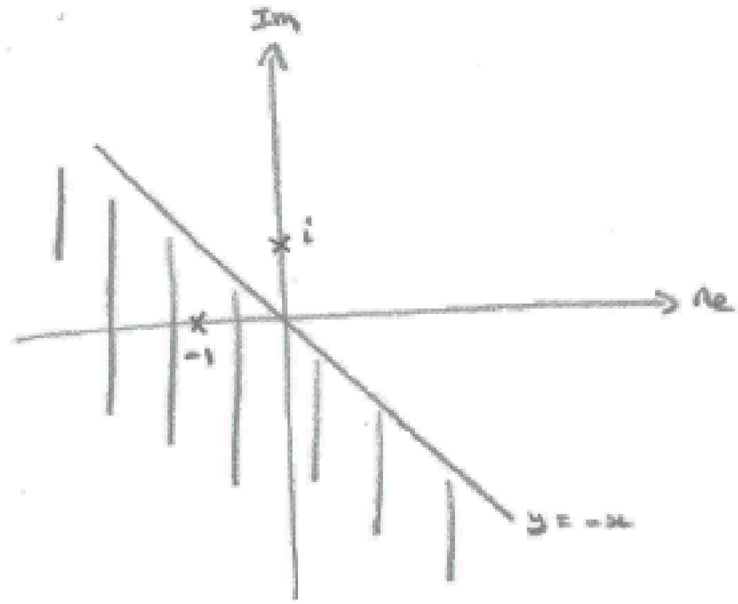
Then $|z - i| > |z + 1|$

$\Rightarrow |x + (y-1)i|^2 > |(x+1) + yi|^2$

$\Rightarrow x^2 + (y-1)^2 > (x+1)^2 + y^2$

$\Rightarrow -2y > 2x$

$\Rightarrow y < -x$



(ii) Let $z = x + yi$

Then $|z - i| = 2|z + 1|$

$\Rightarrow |x + (y-1)i|^2 = 4|(x+1) + yi|^2$

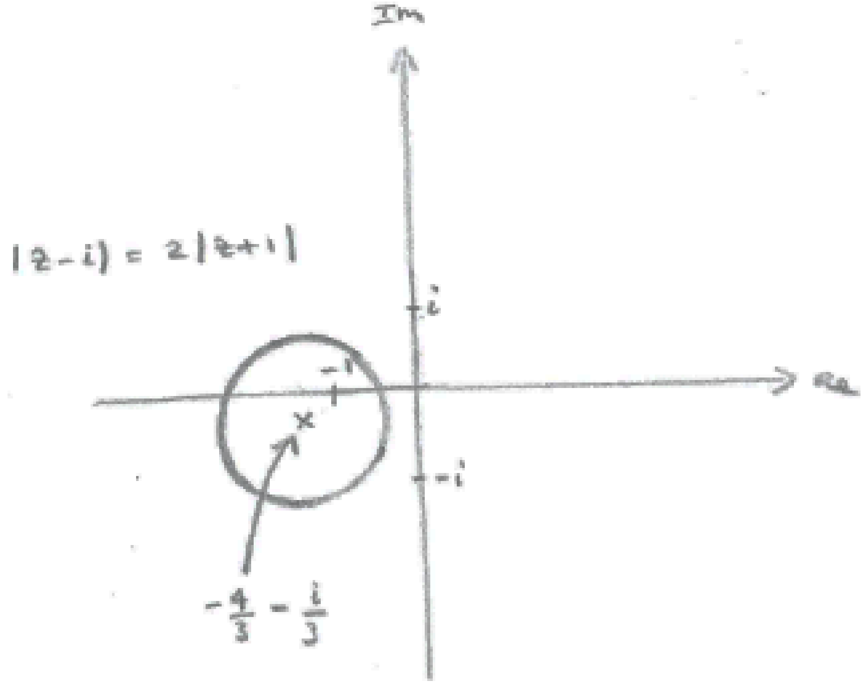
$\Rightarrow x^2 + (y-1)^2 = 4\{(x+1)^2 + y^2\}$

$\Rightarrow 3x^2 + 8x + 3y^2 + 2y + 3 = 0$

$\Rightarrow x^2 + \frac{8x}{3} + y^2 + \frac{2y}{3} + 1 = 0$

$\Rightarrow (x + \frac{4}{3})^2 + (y + \frac{1}{3})^2 - \frac{16}{9} - \frac{1}{9} + 1 = 0$

$\Rightarrow (x + \frac{4}{3})^2 + (y + \frac{1}{3})^2 = \frac{8}{9}$



(5*) $1+3i$ is a root of the equation $z^3+pz+q=0$ (where p & q are real). Find the other roots, and the values of p & q

Solution

As the coefficients of the equation are real, $1-3i$ will also be a root.

Then the equation can be written as

$(z-[1+3i])(z-[1-3i])(z-\alpha)=0$, where α is the 3rd root.

Expanding this gives $(z^2-2z+10)(z-\alpha)=0$

and hence $z^3-(2+\alpha)z^2+(10+2\alpha)z-10\alpha=0$

Comparing the coefficients with those of $z^3+pz+q=0$,

we see that $\alpha=-2$, so that $p=6$ and $q=20$

Alternative method

Using the standard results that the roots α, β & γ of the equation

$az^3 + bz^2 + cz + d = 0$ satisfy $\alpha + \beta + \gamma = -\frac{b}{a}$, $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$ and $\alpha\beta\gamma = -\frac{d}{a}$ (*):

$(1 + 3i) + (1 - 3i) + \alpha = 0$ [since $b = 0$]

Hence $\alpha = -2$

Also $(1 + 3i)(1 - 3i) - 2(1 + 3i) - 2(1 - 3i) = p$,

so that $10 - 2 - 2 = p$ and $p = 6$

And $-2(1 + 3i)(1 - 3i) = -q$,

so that $q = 2(10) = 20$

Notes

(a) A cubic function $y = f(x)$ **with real coefficients** will cross the x-axis at least once, and so $f(x) = 0$ has at least one real root (α, say). Then, factorising $f(x)$ as $(x - \alpha)g(x)$ means that, if β is a complex root of $f(x) = 0$, then β^*, the complex conjugate of β, must be the other root (considering the two roots derived from the quadratic formula).

[This could also have been written as $y = f(z)$ etc]

(b) (*) follows from expanding $(z - \alpha)(z - \beta)(z - \gamma) = 0$, and is in fact true whether the coefficients a, b & c are real or complex

(6*) Find the square roots of $3 - 4i$

Solution

We need to find z such that $z^2 = 3 - 4i$

Let $z = a + bi$

Then $a^2 - b^2 + 2abi = 3 - 4i$

Equating real and imaginary parts, $a^2 - b^2 = 3$ and $2ab = -4$

Hence $b = -\frac{2}{a}$ and $a^2 - \frac{4}{a^2} = 3$, so that $a^4 - 3a^2 - 4 = 0$

Then $(a^2 - 4)(a^2 + 1) = 0$

As a is real, $a = \pm 2$ and $b = \mp 1$

Thus the square roots are $2 - i$ and $-2 + i$ or $\pm(2 - i)$

(7***) Let $z = \frac{a+i}{1+ai}$. If $arg z = -\frac{\pi}{4}$, find the possible values of a

Solution

z can be written as $x - xi$, where $x > 0$,

so that $(x - xi)(1 + ai) = a + i$

and $x + xai - xi + xa = a + i$

Then equating real and imaginary parts:

$x + xa = a$ & $xa - x = 1$;

ie $x(1 + a) = a$ & $x(a - 1) = 1$,

so that $x = \frac{a}{1+a} = \frac{1}{a-1}$

and $a^2 - a = 1 + a$

$\Rightarrow a^2 - 2a - 1 = 0$

$\Rightarrow a = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$

Also $x > 0$:

$a = 1 \pm \sqrt{2} \Rightarrow x = \frac{1}{a-1} = \frac{1}{\pm\sqrt{2}}$

so that $a = 1 + \sqrt{2}$

(8*) For each of the following numbers, say whether they are imaginary or complex (or both):

(i) 1 (ii) i (iii) 0 (iv) $1 + i$

Solution

All four are complex (as they appear somewhere in the Argand diagram). Only the numbers i and 0 are imaginary (as they appear on the imaginary axis).

Imaginary numbers are sometimes referred to as "pure imaginary", to avoid confusion.

[$1 + i$ can be described as "non-real complex", to distinguish it from "real and complex" numbers such as 1]

(9*) Are these statements true or false? (Give an explanation, or a counter example, as appropriate.)

(i) All imaginary numbers are complex numbers.

(ii) All complex numbers are imaginary numbers.

(iii) All real numbers are complex numbers.

(iv) Zero is an imaginary number.

(v) The imaginary part of a complex number is an imaginary number.

(vi) All complex numbers are either real numbers or imaginary numbers.

(vii) Two imaginary numbers added together can sometimes give a real number.

(viii) If two complex numbers multiply to give a real number, then they must be conjugates of each other.

(ix) The square root of a non-real complex number is never real.

Solution

(i) True: An imaginary number is a number of the form bi, where b is real; a complex number is a number of the form $a + bi$, where a & b are real, and a can equal zero. Note: "imaginary" numbers are often referred to as "pure imaginary" numbers, to avoid confusion.

(ii) False: The complex number $a + bi$, where $a \neq 0$ is not imaginary, by the definition in (i).

(iii) True: $a + 0i$ is complex.

(iv) True: $0 = 0i$ is imaginary

(v) False: The imaginary part of $a + bi$ is b (not bi : there is an error to this effect in the AQA FP2 website booklet - unless it's been corrected)

(vi) False: $2 + 3i$ is neither real nor imaginary.

(vii) True: For example, i & $-i$

(viii) False: For example, i & i

(ix) True: Suppose that $\sqrt{a + bi} = c$, where $a, b \neq 0$ & c are real;

then $a + bi = c^2$, and equating imaginary parts $\Rightarrow b = 0$, which is a contradiction

(10*) How are the complex numbers z and zi related?

Solution

$|i| = 1$ & $arg(i) = \frac{\pi}{2}$; hence multiplication by i has the effect of rotating z by $\frac{\pi}{2}$ anti-clockwise.

(11***) Find the solutions of $z^2 = i$ by

(a) setting $z = a + bi$ and equating real and imaginary parts

(b) using de Moivre's theorem

Solution

(a) Let $\sqrt{i} = a + bi$

Then $i = (a + bi)^2 = a^2 - b^2 + 2abi$

Equating real & imaginary parts,

$2ab = 1$ (1) & $a^2 - b^2 = 0$ (2)

$\Rightarrow a^2 - \left(\frac{1}{2a}\right)^2 = 0$

$\Rightarrow \left(a - \frac{1}{2a}\right)\left(a + \frac{1}{2a}\right) = 0$

$\Rightarrow$ either $a = \frac{1}{2a} \Rightarrow a^2 = \frac{1}{2} \Rightarrow a = \pm\frac{1}{\sqrt{2}}$

or $a = -\frac{1}{2a} \Rightarrow a^2 = -\frac{1}{2}$ (not possible, as a is real)

Then $a = +\frac{1}{\sqrt{2}} \Rightarrow b = \frac{1}{2a} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$, from (1)

and $a = -\frac{1}{\sqrt{2}} \Rightarrow b = -\frac{1}{\sqrt{2}}$

Thus $\sqrt{i} = \pm\frac{1}{\sqrt{2}}(1+i)$

(This can be checked by squaring the RHS.)

(b) $z^2 = i = \cos\left(\frac{\pi}{2}\right) + isin(\frac{\pi}{2})$

By De Moivre's theorem, $z = \cos\left(\frac{\pi}{4}\right) + isin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(1+i)$

or $z = \cos\left(\frac{\pi}{4} + \frac{(-2\pi)}{2}\right) + isin\left(\frac{\pi}{4} + \frac{(-2\pi)}{2}\right)$

$= \cos\left(-\frac{3\pi}{4}\right) + isin(-\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}(1+i)$

[Note that $\frac{\pi}{4} + \frac{(-2\pi)}{2}$ is chosen as the argument of the 2nd root, rather than $\frac{\pi}{4} + \frac{2\pi}{2}$, to avoid having to subtract 2π at the end.]

(12*) Simplify $e^{i\pi} + 1$

Solution

$\arg\left(e^{i\pi}\right) = \pi$ and $\left|e^{i\pi}\right| = 1$,

so $e^{i\pi} = -1$, and $e^{i\pi} + 1 = 0$

(13*) How are the complex numbers z and $\frac{1}{z}$ related to each other?

Solution

$\left|\frac{1}{z}\right| = \frac{1}{|z|}$ and $arg\left(\frac{1}{z}\right) = arg(1) - arg(z) = -arg(z)$

When $|z| = 1$, z can be written as $cos\theta + isin\theta$, so that

$\frac{1}{z} = cos(-\theta) + isin(-\theta) = cos\theta - isin\theta = z^*$

(14**) Find $(1+i)^{10}$ by considering rotations and magnifications in the Argand diagram

Solution

$\arg(1+i)=\frac{\pi}{4}$ & $|1+i|=\sqrt{2}$

So $(1+i)^2=2e^{2\left(\frac{\pi}{4}\right)i}=2e^{\frac{\pi i}{2}}=2i$

Then multiplication by $(1+i)^8$ results in a magnification of $\left(\sqrt{2}\right)^8=16$ and rotation of $8\left(\frac{\pi}{4}\right)=2\pi$; ie no change

So $(1+i)^{10}=(2i)(16)=32i$

[Or $(1+i)^{10}=\left(\sqrt{2}e^{\frac{\pi i}{4}}\right)^{10}=32e^{\frac{10\pi i}{4}}=32e^{\frac{5\pi i}{2}}=32e^{\frac{\pi i}{2}}=32i$]

(15*) Show that, if ω is an nth root of unity, then ω^r is also (where n & r are positive integers).

Solution

$(\omega^r)^n=\omega^{rn}=(\omega^n)^r=1^r=1$

(16**) Find the equation of the line satisfying

$|z+10|=|z-6-4i\sqrt{2}|$

Solution

Squaring both sides, $(x+10)^2+y^2=(x-6)^2+(y-4\sqrt{2})^2$

$\Rightarrow 20x+100=-12x+36-8\sqrt{2}y+32$

$\Rightarrow 8\sqrt{2}y=-32x-32$

$\Rightarrow y=-2\sqrt{2}x-2\sqrt{2}$

(17**) Find $\arg\{-sin\left(\frac{\pi}{3}\right) + icos\left(\frac{\pi}{3}\right)\}$, other than by just plotting the point in the Argand diagram.

Solution

Approach 1

$-sin\left(\frac{\pi}{3}\right) + icos\left(\frac{\pi}{3}\right) = sin\left(-\frac{\pi}{3}\right) + icos\left(-\frac{\pi}{3}\right)$

[note that it helps to keep the angle the same in both terms]

$= cos\left(\frac{\pi}{2} - [-\frac{\pi}{3}]\right) + isin\left(\frac{\pi}{2} - [-\frac{\pi}{3}]\right) = \cos\left(\frac{5\pi}{6}\right) + isin(\frac{5\pi}{6})$

So $\arg\{-sin\left(\frac{\pi}{3}\right) + icos\left(\frac{\pi}{3}\right)\} = \frac{5\pi}{6}$

Approach 2

$-sin\left(\frac{\pi}{3}\right) + icos\left(\frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) + isin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$

$= -cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right) = -\{cos\left(\frac{\pi}{6}\right) - isin\left(\frac{\pi}{6}\right)\}$

Then $arg\left\{cos\left(\frac{\pi}{6}\right) - isin\left(\frac{\pi}{6}\right)\right\} = -\frac{\pi}{6}$

[as $cos\left(\frac{\pi}{6}\right) - isin\left(\frac{\pi}{6}\right)$ is the conjugate of $cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right)$;

also $cos\left(\frac{\pi}{6}\right) - isin\left(\frac{\pi}{6}\right) = cos\left(-\frac{\pi}{6}\right) + isin\left(-\frac{\pi}{6}\right)$],

and so $\arg\left[-\left\{cos\left(\frac{\pi}{6}\right) - isin\left(\frac{\pi}{6}\right)\right\}\right] = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$

[since multiplication by -1 is a rotation by π in the Argand diagram]

(18***) Find the mod and arg of $e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}}$

Solution

Method 1

Write $z = e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}}$ in the form $e^{a\pi i}(e^{b\pi i} - e^{-b\pi i})$

So $a + b = \frac{7}{10}$ & $a - b = -\frac{9}{10}$

Then $a = -\frac{1}{10}$ & $b = \frac{8}{10}$

and $e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}} = e^{-\frac{\pi i}{10}}(e^{\frac{8\pi i}{10}} - e^{-\frac{8\pi i}{10}})$

$= e^{-\frac{\pi i}{10}}(2isin\left(\frac{4\pi}{5}\right))$

Then $|z| = \left|e^{-\frac{\pi i}{10}}\right| \left|2isin\left(\frac{4\pi}{5}\right)\right| = (1)(2sin\left(\frac{4\pi}{5}\right))$

$= 2\sin\left(\pi - \frac{4\pi}{5}\right) = 2\sin(\frac{\pi}{5})$

and $\arg(z) = \arg\left(e^{-\frac{\pi i}{10}}\right) + \arg(2isin\left(\frac{4\pi}{5}\right))$

$= -\frac{\pi}{10} + \frac{\pi}{2} = \frac{4\pi}{10} = \frac{2\pi}{5}$

Method 2

$e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}}$

$= \left(cos\left(\frac{7\pi}{10}\right) - cos\left(\frac{-9\pi}{10}\right)\right) + i\left(sin\left(\frac{7\pi}{10}\right) - sin\left(\frac{-9\pi}{10}\right)\right)$

$= -2sin\left(\frac{1}{2}(\frac{7\pi}{10} + \frac{-9\pi}{10})\right) sin\left(\frac{1}{2}(\frac{7\pi}{10} - \frac{-9\pi}{10})\right)$

$+2cos\left(\frac{1}{2}(\frac{7\pi}{10} + \frac{-9\pi}{10})\right) sin\left(\frac{1}{2}(\frac{7\pi}{10} - \frac{-9\pi}{10})\right)$

$= -2sin\left(-\frac{\pi}{10}\right)sin\left(\frac{8\pi}{10}\right) + 2icos\left(-\frac{\pi}{10}\right)sin\left(\frac{8\pi}{10}\right)$

$= 2sin\left(\frac{8\pi}{10}\right)\{sin\left(\frac{\pi}{10}\right) + icos\left(\frac{\pi}{10}\right)\}$

$= 2sin\left(\frac{4\pi}{5}\right)\{cos\left(\frac{\pi}{2} - \frac{\pi}{10}\right) + isin\left(\frac{\pi}{2} - \frac{\pi}{10}\right)\}$

$= 2sin\left(\frac{\pi}{5}\right)\{cos\left(\frac{4\pi}{10}\right) + isin\left(\frac{4\pi}{10}\right)\}$

$= 2sin\left(\frac{\pi}{5}\right)e^{\frac{2\pi i}{5}}$

So mod is $2sin\left(\frac{\pi}{5}\right)$ and arg is $\frac{2\pi}{5}$

(19**) Find i^i in cartesian form (ie $x + yi$)

Solution

$i^i = \left(e^{i(\frac{\pi}{2}+2k\pi)}\right)^i = e^{-(\frac{\pi}{2}+2k\pi)}$ for $k \in \mathbb{Z}$

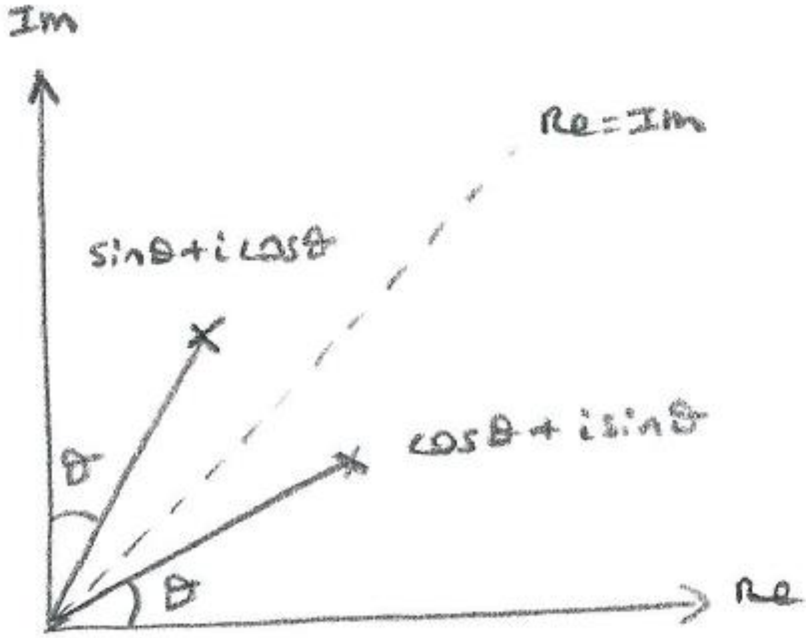
(ie i^i is a collection of real numbers)

(20**) How are the complex numbers $cos\theta + isin\theta$ and $sin\theta + icos\theta$ related?

Solution

$sin\theta + icos\theta = \cos\left(\frac{\pi}{2} - \theta\right) + isin(\frac{\pi}{2} - \theta)$

As both complex numbers have a modulus of 1, $sin\theta + icos\theta$ is the reflection of $cos\theta + isin\theta$ in the line Re = Im (see diagram below).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Re=Im
sinθ+i cosθ
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