# **Complex Numbers - Exercises** (45 pages; 22/3/25)

Represent the following on the Argand diagram:

(i) $|z-i|>|z+1|$

(ii) $|z-i|=2|z+1|$

[Represent the following on the Argand diagram:

(i) $|z - i| > |z + 1|$

(ii) $|z - i| = 2|z + 1|$ ]

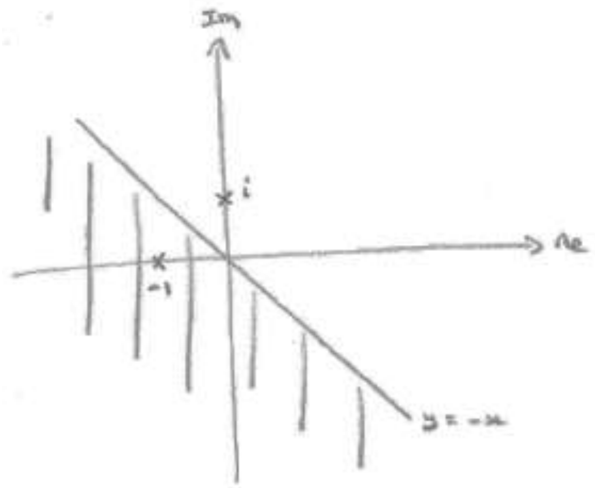
**Solution**

(i) **Method 1**

Rewriting as $|z - i| > |z - (-1)|$ ,

$z$ has to be further from $i$ than from $-1$;

When $z$ is equidistant from these two points, it lies on the perpendicular bisector of the line (segment) connecting the points. So the required region is as shown below.




**Method 2**

Let $z = x + yi$

Then $|z - i| > |z + 1|$

$\Rightarrow |x + (y - 1)i|^2 > |(x + 1) + yi|^2$

$\Rightarrow x^2 + (y - 1)^2 > (x + 1)^2 + y^2$

$\Rightarrow -2y > 2x$

$\Rightarrow y < -x$

(ii) Let $z = x + yi$

Then $|z - i| = 2|z + 1|$

$\Rightarrow |x + (y-1)i|^2 = 4|(x+1) + yi|^2$

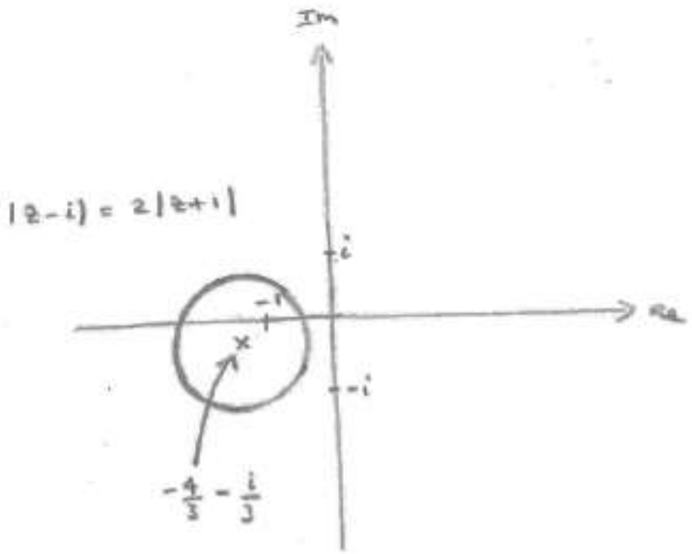
$\Rightarrow x^2 + (y-1)^2 = 4\{(x+1)^2 + y^2\}$

$\Rightarrow 3x^2 + 8x + 3y^2 + 2y + 3 = 0$

$\Rightarrow x^2 + \frac{8x}{3} + y^2 + \frac{2y}{3} + 1 = 0$

$\Rightarrow (x + \frac{4}{3})^2 + (y + \frac{1}{3})^2 - \frac{16}{9} - \frac{1}{9} + 1 = 0$

$\Rightarrow (x + \frac{4}{3})^2 + (y + \frac{1}{3})^2 = \frac{8}{9}$



$1+3i$ is a root of the equation $z^3+pz+q=0$ (where p & q are real). Find the other roots, and the values of p & q.

[$1+3i$ is a root of the equation $z^3 + pz + q = 0$ (where p & q are real). Find the other roots, and the values of p & q.]

**Solution**

As the coefficients of the equation are real, the conjugate of

$1+3i$: $1-3i$ will also be a root.

Then the equation can be written as

$(z-[1+3i])(z-[1-3i])(z-\alpha)=0$ , where $\alpha$ is the 3rd root.

Expanding this gives $(z^2-2z+10)(z-\alpha)=0$

and hence $z^3-(2+\alpha)z^2+(10+2\alpha)z-10\alpha=0$

Comparing the coefficients with those of $z^3+pz+q=0$,

we see that $\alpha=-2$, so that $p=6$ and $q=20$

## Alternative method

Using the standard results that the roots $\alpha, \beta$ & $\gamma$ of the equation $az^3 + bz^2 + cz + d = 0$ satisfy $\alpha + \beta + \gamma = -\frac{b}{a}$, $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$ and $\alpha\beta\gamma = -\frac{d}{a}$ (*):

$(1 + 3i) + (1 - 3i) + \alpha = 0$ [since $b = 0$]

Hence $\alpha = -2$

Also $(1 + 3i)(1 - 3i) - 2(1 + 3i) - 2(1 - 3i) = p$,

so that $10 - 2 - 2 = p$ and $p = 6$

And $-2(1 + 3i)(1 - 3i) = -q$,

so that $q = 2(10) = 20$

## Notes

(a) A cubic function $y = f(x)$ **with real coefficients** will cross the $x$-axis at least once, and so $f(x) = 0$ has at least one real root ($\alpha$, say). Then, factorising $f(x)$ as $(x - \alpha)g(x)$ means that, if $\beta$ is a complex root of $f(x) = 0$, then $\beta^*$, the complex conjugate of $\beta$, must be the other root (considering the two roots derived from the quadratic formula).

[This could also have been written as $y = f(z)$ etc]

(b) (*) follows from expanding $(z - \alpha)(z - \beta)(z - \gamma) = 0$, and is in fact true whether the coefficients a, b & c are real or complex.

Find the square roots of $3 - 4i$

[Find the square roots of $3-4i$]

**Solution**

We need to find z such that $z^2=3-4i$

Let $z=a+bi$

Then $a^2-b^2+2abi=3-4i$

Equating real and imaginary parts, $a^2-b^2=3$ and $2ab=-4$

Hence $b=-\frac{2}{a}$ and $a^2-\frac{4}{a^2}=3$, so that $a^4-3a^2-4=0$

Then $(a^2-4)(a^2+1)=0$

As $a$ is real, $a=\pm2$ and $b=\mp1$

Thus the square roots are $2-i$ and $-2+i$ or $\pm(2-i)$

Find the solutions of $z^2 = i$

[Find the solutions of $z^2 = i$ ]

**Solution**

**Method 1**

$z^2 = i = \cos\left(\frac{\pi}{2}\right) + isin(\frac{\pi}{2})$

By De Moivre's theorem, $z = \cos\left(\frac{\pi}{4}\right) + isin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(1 + i)$

or $z = \cos\left(\frac{\pi}{4} + \frac{(-2\pi)}{2}\right) + isin\left(\frac{\pi}{4} + \frac{(-2\pi)}{2}\right)$

$= \cos\left(-\frac{3\pi}{4}\right) + isin(-\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}(1 + i)$

[Note that $\frac{\pi}{4} + \frac{(-2\pi)}{2}$ is chosen as the argument of the 2nd root, rather than $\frac{\pi}{4} + \frac{2\pi}{2}$, to avoid having to subtract $2\pi$ at the end.]

**Method 2**

Let $\sqrt{i} = a + bi$

Then $i = (a + bi)^2 = a^2 - b^2 + 2abi$

Equating real & imaginary parts,

$2ab = 1$ (1) & $a^2 - b^2 = 0$ (2)

$\Rightarrow a^2 - \left(\frac{1}{2a}\right)^2 = 0$

$\Rightarrow \left(a - \frac{1}{2a}\right)\left(a + \frac{1}{2a}\right) = 0$

$\Rightarrow$ either $a = \frac{1}{2a} \Rightarrow a^2 = \frac{1}{2} \Rightarrow a = \pm\frac{1}{\sqrt{2}}$

or $a = -\frac{1}{2a} \Rightarrow a^2 = -\frac{1}{2}$ (not possible, as $a$ is real)

Then $a = +\frac{1}{\sqrt{2}} \Rightarrow b = \frac{1}{2a} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ , from (1) and $a = -\frac{1}{\sqrt{2}} \Rightarrow b = -\frac{1}{\sqrt{2}}$

Thus $\sqrt{i} = \pm\frac{1}{\sqrt{2}}(1 + i)$

(This can be checked by squaring the RHS.)

Find $(1+i)^{10}$

[Find $(1+i)^{10}$]

**Solution**

$\arg(1+i) = \frac{\pi}{4}$ & $|1+i| = \sqrt{2}$

So $(1+i)^2 = 2e^{2\left(\frac{\pi}{4}\right)i} = 2e^{\frac{\pi i}{2}} = 2i$

Then multiplication by $(1+i)^8$ results in a magnification of $\left(\sqrt{2}\right)^8 = 16$ and rotation of $8\left(\frac{\pi}{4}\right) = 2\pi$; ie no change

So $(1+i)^{10} = (2i)(16) = 32i$

[Or $(1+i)^{10} = \left(\sqrt{2}e^{\frac{\pi i}{4}}\right)^{10} = 32e^{\frac{10\pi i}{4}} = 32e^{\frac{5\pi i}{2}} = 32e^{\frac{\pi i}{2}} = 32i$]

Find the equation of the line satisfying

$|z + 10| = |z - 6 - 4i\sqrt{2}|$

[Find the equation of the line satisfying

$|z+10| = |z-6-4i\sqrt{2}|$ ]

**Solution**

Writing $z = x + yi$,

$|z+10| = \left|z-6-4i\sqrt{2}\right| \Rightarrow |x+10+yi| = \left|x-6+\left(y-4\sqrt{2}\right)i\right|$

Squaring both sides, $(x+10)^2 + y^2 = (x-6)^2 + (y-4\sqrt{2})^2$

$\Rightarrow 20x + 100 = -12x + 36 - 8\sqrt{2}y + 32$

$\Rightarrow 8\sqrt{2}y = -32x - 32$

$\Rightarrow y = -2\sqrt{2}x - 2\sqrt{2}$

Express $(1 - i)^6$ in the form $x + iy$

[Express $(1-i)^6$ in the form $x+iy$ ]

**Solution**

First of all, express $z = 1 - i$ in modulus-argument form:

By considering the Argand diagram, $|z| = \sqrt{2}$ & $\arg(z) = -\frac{\pi}{4}$

So $z = \sqrt{2}(\cos\left(-\frac{\pi}{4}\right) + isin\left(-\frac{\pi}{4}\right))$

Then, by de Moivre's theorem,

$$z^6 = \left(\sqrt{2}\right)^6\left(\cos\left(-\frac{6\pi}{4}\right) + isin(-\frac{6\pi}{4})\right)$$

$$= 8\left(\cos\left(-\frac{3\pi}{2}\right) + isin\left(-\frac{3\pi}{2}\right)\right)$$

$$= 8\left(\cos\left(\frac{\pi}{2}\right) + isin\left(\frac{\pi}{2}\right)\right) = 8i$$

Find the cube roots of $-8$ in cartesian form

[Find the cube roots of $-8$ in cartesian form]

**Solution**

$z^3 = 8(cos\pi + isin\pi)$

$z_1 = 2(\cos\left(\frac{\pi}{3}\right) + isin\left(\frac{\pi}{3}\right))$

$z_2 = 2\left(\cos\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) + isin\left(\frac{\pi}{3} + \frac{2\pi}{3}\right)\right) = 2(\cos\pi + i\sin\pi)$

$z_3 = 2(\cos\left(\frac{\pi}{3} + \frac{4\pi}{3}\right) + isin\left(\frac{\pi}{3} + \frac{4\pi}{3}\right)) = 2(\cos\left(\frac{5\pi}{3}\right) + isin\left(\frac{5\pi}{3}\right))$

$= 2(\cos\left(-\frac{\pi}{3}\right) + isin\left(-\frac{\pi}{3}\right))$

ie $z_1 = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + \sqrt{3}\,i$

$z_2 = -2$

$z_3 = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}\,i$ [$z^3 + 8 = 0 \Rightarrow {z_1}^*$ is root]

For each of the following numbers, say whether they are imaginary or complex (or both):

(i) $1$ (ii) $i$ (iii) $0$ (iv) $1+i$

[For each of the following numbers, say whether they are imaginary or complex (or both):

(i) 1  (ii) $i$  (iii) 0  (iv) $1 + i$ ]

**Solution**

All four are complex (as they appear somewhere in the Argand diagram). Only the numbers $i$ and 0 are imaginary (as they appear on the imaginary axis).

Imaginary numbers are sometimes referred to as "pure imaginary", to avoid confusion.

[$1 + i$ can be described as "non-real complex", to distinguish it from "real and complex" numbers such as 1]

Are these statements true or false? (Give an explanation, or a counter example, as appropriate.)

(i) All imaginary numbers are complex numbers.

(ii) All complex numbers are imaginary numbers.

(iii) All real numbers are complex numbers.

(iv) Zero is an imaginary number.

(v) The imaginary part of a complex number is an imaginary number.

(vi) All complex numbers are either real numbers or imaginary numbers.

(vii) Two imaginary numbers added together can sometimes give a real number.

(viii) If two complex numbers multiply to give a real number, then they must be conjugates of each other.

(ix) The square root of a non-real complex number is never real.

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(ix) The square root of a non-real complex number is never real.]

## Solution

(i) True: An imaginary number is a number of the form $bi$, where $b$ is real; a complex number is a number of the form $a + bi$, where $a$ & $b$ are real, and $a$ can equal zero. Note: "imaginary" numbers are often referred to as "pure imaginary" numbers, to avoid confusion.

(ii) False: The complex number $a + bi$, where $a \neq 0$ is not imaginary, by the definition in (i).

(iii) True: $a + 0i$ is complex.

(iv) True: $0 = 0i$ is imaginary

(v) False: The imaginary part of $a + bi$ is $b$

(vi) False: $2 + 3i$ is neither real nor imaginary.

(vii) True: For example, $i$ & $-i$

(viii) False: For example, $i$ & $i$

(ix) True: Suppose that $\sqrt{a + bi} = c$, where $a, b \neq 0$ & $c$ are real;

then $a + bi = c^2$, and equating imaginary parts $\Rightarrow b = 0$, which is a contradiction

Find $(2+5i) \div (1+3i)$ by (a) equating real and imaginary parts, and (b) another method.

[Find $(2+5i) \div (1+3i)$ by (a) equating real and imaginary parts, and (b) another method]

**Solution**

(a) Let $(2+5i) \div (1+3i) = a + bi$

Then $2 + 5i = (a+bi)(1+3i) = a + 3ai + bi - 3b$

Equating real parts: $2 = a - 3b$ (1)

Equating imaginary parts: $5 = 3a + b$ (2)

$(1) + 3 \times (2) \Rightarrow 17 = 10a \Rightarrow a = \frac{17}{10}$

Then $(2) \Rightarrow b = 5 - \frac{51}{10} = -\frac{1}{10}$

So $(2+5i) \div (1+3i) = \frac{17}{10} - \frac{i}{10}$

(b) $\frac{2+5i}{1+3i} = \frac{(2+5i)(1-3i)}{(1+3i)(1-3i)} = \frac{2+15-6i+5i}{1+9} = \frac{17}{10} - \frac{i}{10}$

Check: $\frac{1}{10}$

Solve the equation $(2+i)z+3=0$ by (a) equating real and imaginary parts, and (b) another method

[Solve the equation $(2+i)z+3=0$ by (a) equating real and imaginary parts, and (b) another method]

**Solution**

(a) Let $z = a + bi$

Then $(2+i)(a+bi)+3=0$

$\Rightarrow 2a - b + (a+2b)i + 3 = 0$

Equating real parts: $2a - b = -3$ (1)

Equating imaginary parts: $a + 2b = 0$ (2)

Substituting for a from (2) into (1), $2(-2b) - b = -3$ and $\therefore b = \frac{3}{5}$ and $a = -\frac{6}{5}$

so that $z = -\frac{6}{5} + \frac{3i}{5}$

(b) $(2+i)z+3=0 \Rightarrow z = \frac{-3}{2+i} = \frac{-3(2-i)}{(2+i)(2-i)} = \frac{-6+3i}{4+1} = -\frac{6}{5} + \frac{3i}{5}$

Solve the equation $z^2 - 2z + 2 = 0$

(a) by completing the square
(b) by equating real & imaginary parts

[Solve the equation $z^2 - 2z + 2 = 0$

(a) by completing the square
(b) by equating real & imaginary parts]

**Solution**

(a) $z^2 - 2z + 2 = 0$

$\Rightarrow (z-1)^2 + 1^2 = 0$

$\Rightarrow ([z-1]+i)([z-1]-i) = 0$

$\Rightarrow z = 1 - i \;\; or \;\; 1 + i$

(b) Let $z = a + bi$

Then $(a+bi)^2 - 2(a+bi) + 2 = 0$

$\Rightarrow a^2 - b^2 + 2abi - 2a - 2bi + 2 = 0$

equating real parts: $a^2 - b^2 - 2a + 2 = 0$ (1)

equating imaginary parts: $2ab - 2b = 0$ (2)

(2) $\Rightarrow b(a-1) = 0 \Rightarrow b = 0 \;\; or \;\; a = 1$

From (1), $b = 0 \Rightarrow a^2 - 2a + 2 = 0$

(this can be excluded, as $a$ is real and there are no real solutions to the quadratic equation)

$a = 1 \Rightarrow 1 - b^2 = 0 \Rightarrow b = \pm 1$

Hence $z = 1 \pm i$

Find arg $\{-sin\left(\frac{\pi}{3}\right) + icos\left(\frac{\pi}{3}\right)\}$

[Find arg $\{-sin\left(\frac{\pi}{3}\right) + icos\left(\frac{\pi}{3}\right)\}$ ]

**Solution**

**Approach 1**

$-sin\left(\frac{\pi}{3}\right) + icos\left(\frac{\pi}{3}\right) = sin\left(-\frac{\pi}{3}\right) + icos\left(-\frac{\pi}{3}\right)$

[note that it helps to keep the angle the same in both terms]

$= cos\left(\frac{\pi}{2} - [-\frac{\pi}{3}]\right) + isin\left(\frac{\pi}{2} - [-\frac{\pi}{3}]\right) = \cos\left(\frac{5\pi}{6}\right) + isin(\frac{5\pi}{6})$

So arg $\{-sin\left(\frac{\pi}{3}\right) + icos\left(\frac{\pi}{3}\right)\} = \frac{5\pi}{6}$

**Approach 2**

$-sin\left(\frac{\pi}{3}\right) + icos\left(\frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) + isin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$

$= -cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right) = -\{cos\left(\frac{\pi}{6}\right) - isin\left(\frac{\pi}{6}\right)\}$

Then $arg\left\{cos\left(\frac{\pi}{6}\right) - isin\left(\frac{\pi}{6}\right)\right\} = -\frac{\pi}{6}$

[as $cos\left(\frac{\pi}{6}\right) - isin\left(\frac{\pi}{6}\right)$ is the conjugate of $cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right)$;

also $cos\left(\frac{\pi}{6}\right) - isin\left(\frac{\pi}{6}\right) = cos\left(-\frac{\pi}{6}\right) + isin\left(-\frac{\pi}{6}\right)$],

and so arg $\left[-\left\{cos\left(\frac{\pi}{6}\right) - isin\left(\frac{\pi}{6}\right)\right\}\right] = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$

[since multiplication by $-1$ is a rotation by $\pi$ in the Argand diagram]

**Approach 3**

arg $\left\{-sin\left(\frac{\pi}{3}\right) + icos\left(\frac{\pi}{3}\right)\right\} = $ arg $\{i(cos\left(\frac{\pi}{3}\right) + isin\left(\frac{\pi}{3}\right))\}$

$= arg(i) + \frac{\pi}{3} = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$

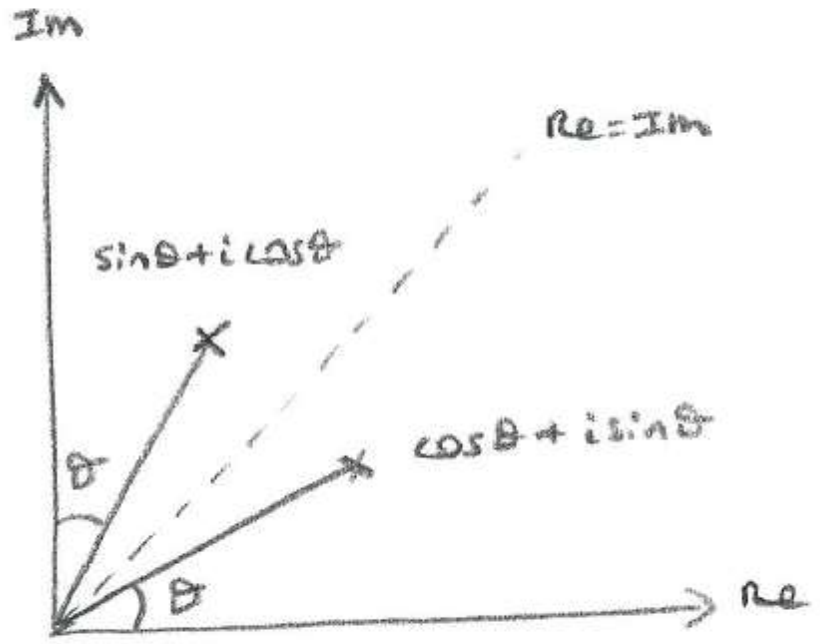
How are the complex numbers $cos\theta + isin\theta$ and $sin\theta + icos\theta$ related?

[How are the complex numbers $cos\theta + isin\theta$ and $sin\theta + icos\theta$ related?]

**Solution**

$sin\theta + icos\theta = \cos\left(\frac{\pi}{2} - \theta\right) + isin(\frac{\pi}{2} - \theta)$

As both complex numbers have a modulus of 1, $sin\theta + icos\theta$ is the reflection of $cos\theta + isin\theta$ in the line Re = Im (see diagram below).



Given that $2 - i$ is a root of the equation

$z^4 - 6z^3 - 2z^2 + 50z - 75 = 0$, find the other roots.

[Given that $2-i$ is a root of the equation

$z^4-6z^3-2z^2+50z-75=0$, find the other roots.]

**Solution**

**Method 1**

$2+i$ is another root (the conjugate of $2-i$)

Let the other two roots be $\alpha$ & $\beta$.

Then $(2-i)+(2+i)+\alpha+\beta=6$; $\alpha+\beta=2$

And $(2-i)(2+i)\alpha\beta=-75$; $5\alpha\beta=-75$; $\alpha\beta=-15$

So the roots $\alpha$ & $\beta$ satisfy $x^2-2x-15=0$

$\Rightarrow (x-5)(x+3)=0 \Rightarrow x=5\ or-3$, and these are the remaining roots.

**Method 2**

$2+i$ is another root (the conjugate of $2-i$)

Write $z^4-6z^3-2z^2+50z-75$

$=(z-[2-i])(z-[2+i])(z^2+bz+c)$

$=(z^2-4z+5)(z^2+bz+c)$,

as $(2-i)+(2+i)=4$ and $(2-i)(2+i)=2^2+1^2=5$

Then, equating coefficients,

$c=-15$ and $[z^3:]-6=b-4$, so that $b=-2$

[Check: $[z^2:]-2=-15-4b+5 \Rightarrow b=-2$]

Thus $z^4-6z^3-2z^2+50z-75=(z^2-4z+5)(z^2-2z-15)$

And $z^2-2z-15=0 \Rightarrow (z-5)(z+3)=0 \Rightarrow z=5\ or-3$, and these are the remaining roots.

(i) Show geometrically that

$|z_1 + z_2| \leq |z_1| + |z_2|$

When is there equality?

(ii) Show geometrically, and also from (i) that

$|z_1 - z_2| \geq |z_1| - |z_2|$

When is there equality?

[(i) Show geometrically that

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When is there equality?

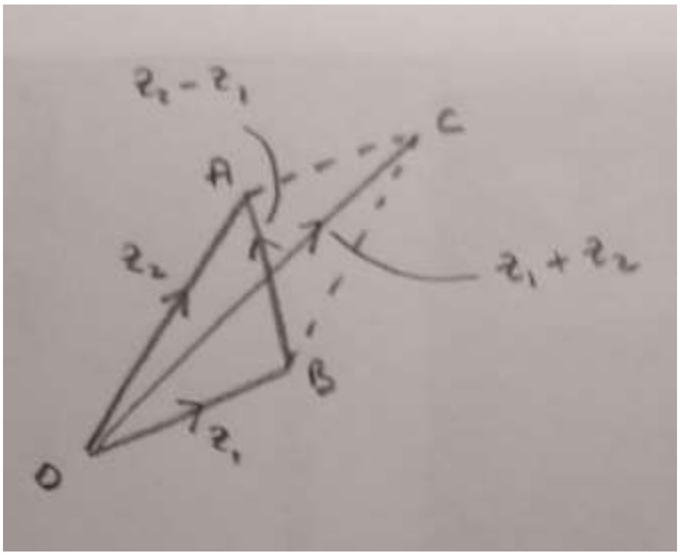
(ii) Show geometrically, and also from (i) that

$|z_1 - z_2| \geq |z_1| - |z_2|$

When is there equality?]

**Solution**

**(i)**




Referring to the diagram, $|z_1 + z_2|$ is the length OC, whilst $|z_1|$ and $|z_2|$ are the lengths AC and OA. As $OC \leq OA + AC$,

the required result follows.

If $z_2 = kz_1$ (so that $z_1$ & $z_2$ have the same argument),

then $|z_1 + z_2| = |(1 + k)z_1| = (1 + k)|z_1|$

and $|z_1| + |z_2| = |z_1| + k|z_1| = (1 + k)|z_1|$

So there is equality when $z_1$ & $z_2$ have the same argument.

[Strictly speaking, we should also show that $|z_1 - z_2| = |z_1| + |z_2|$ means that $z_2 = kz_1$, and this can be seen geometrically, by requiring A to lie on OC.]

(ii) Referring to the diagram again, $|z_1 - z_2| = |z_2 - z_1|$ is the length BA.

Result to prove: $|z_1 - z_2| \geq |z_1| - |z_2|$ ; ie $BA \geq OB - OA$,

or $OB \leq OA + BA$, and this can be seen to be true from the diagram.

Alternatively, from (i): $|z_1 + z_2| \leq |z_1| + |z_2|$

or $|z_1| \geq |z_1 + z_2| - |z_2|$

So let $z_1 = u_1 - u_2$ and $z_2 = u_2$.

Then $|u_1 - u_2| \geq |(u_1 - u_2) + u_2| - |u_2|$

ie $|u_1 - u_2| \geq |u_1| - |u_2|$,

which can be rewritten as $|z_1 - z_2| \geq |z_1| - |z_2|$ , as required.

Equality occurs when $|z_1 - z_2| = |z_1| - |z_2|$ ;

ie $|z_1| = |z_2| + |z_1 - z_2|$,

which is when $|z_1| \geq |z_2|$ and $z_1 = kz_2$, so that $k \geq 1$.

Points representing the 3 roots of the equation

$z^3 + z^2 - 7z - 15 = 0$ are plotted on an Argand diagram.

Given that one of the roots is an integer, find the area of the triangle that has these 3 points as its vertices.

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$z^3 + z^2 - 7z - 15 = 0$ are plotted on an Argand diagram.

Given that one of the roots is an integer, find the area of the triangle that has these 3 points as its vertices.]

**Solution**

Let $f(z) = z^3 + z^2 - 7z - 15$

If $f(z)$ is to factorise, then we need only consider factors of 15 when applying the Factor theorem.

$f(1) = 1 + 1 - 7 - 15 = -20$

$f(-1) = -1 + 1 + 7 - 15 = -8$

$f(3) = 27 + 9 - 21 - 15 = 0$

Thus $z - 3$ is a factor,

and we can write $z^3 + z^2 - 7z - 15 = (z - 3)(z^2 + 4z + 5)$

The roots are therefore 3 & $\frac{-4\pm\sqrt{16-20}}{2} = -2 \pm i$

The area of the triangle is thus $\frac{1}{2}(5)(2) = 5$ sq. units.

Find the square roots of $-5 - 12i$

[Find the square roots of $-5-12i$]

**Solution**

Let $(a+bi)^2=-5-12i$

Then, equating Re. & Im parts:

$a^2-b^2=-5$ & $2ab=-12$

so that $a^2-(\frac{-12}{2a})^2=-5$

$\Rightarrow a^4-36=-5a^2$

Writing $c=a^2, c^2+5c-36=0$

$\Rightarrow (c+9)(c-4)=0$

$\Rightarrow a^2=4$ (reject $a^2=-9$, as negative)

$\Rightarrow a=\pm 2$

$a=2\Rightarrow b=\frac{-12}{2a}=-3$

and $a=-2\Rightarrow b=3$

So the square roots are $2-3i$ and $-2+3i$ [ie $\pm(2-3i)$]

Use complex numbers to show that $sin\left(\frac{5\pi}{12}\right) = \frac{\sqrt{2}+\sqrt{6}}{4}$

[Use complex numbers to show that $sin\left(\frac{5\pi}{12}\right) = \frac{\sqrt{2}+\sqrt{6}}{4}$]

**Solution**

$\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$

Let $z_1 = \cos\left(\frac{\pi}{4}\right) + isin(\frac{\pi}{4})$ and $z_2 = \cos\left(\frac{\pi}{6}\right) + isin(\frac{\pi}{6})$

Then $\arg(z_1z_2) = \arg(z_1) + \arg(z_2) = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$,

and hence $sin\left(\frac{5\pi}{12}\right) = \frac{Im(z_1z_2)}{|z_1z_2|} = \frac{Im(z_1z_2)}{|z_1||z_2|} = Im(z_1z_2)$

Then, as $z_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $z_2 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ ,

$Im(z_1z_2) = \frac{1}{\sqrt{2}} . \frac{1}{2} + \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}$ , as required.