Complex Numbers – Further Exercises

(10 pages; 22/3/25)

Consider two roots of $z^n = cos\theta + isin\theta$:

$z_r = \cos\left(\frac{\theta}{n} + \frac{2\pi r}{n}\right) + isin\left(\frac{\theta}{n} + \frac{2\pi r}{n}\right)$

and $z_R = \cos\left(\frac{\theta}{n} + \frac{2\pi R}{n}\right) + isin\left(\frac{\theta}{n} + \frac{2\pi R}{n}\right)$

(i) Find the condition on n for z_R to equal -z_r for some R, and find R in terms of r.

(ii) Find the condition on θ for z_R to be the conjugate of z_r for some R, and find R in terms of r.

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Solution

(i) $\frac{\theta}{n} + \frac{2\pi R}{n} = \frac{\theta}{n} + \frac{2\pi r}{n} \pm \pi$

$\Rightarrow 2R = 2r \pm n$

$\Rightarrow R = r \pm \frac{n}{2}$

So n has to be even.

(ii) $\frac{\theta}{n} + \frac{2\pi R}{n} = -(\frac{\theta}{n} + \frac{2\pi r}{n})$

$\Rightarrow 2\pi R = -2\theta - 2\pi r$

$\Rightarrow R = -\frac{\theta}{\pi} - r$

So θ has to be either 0 or π (ie a $= cos\theta + isin\theta$ has to be real,

so that $z^n - a = 0$ has real coefficients)

If $\theta = 0$, then $R = -r$, and if $\theta = \pi$, then $R = -1 - r$

Example 1: $z^n = 1 = cos0, +isin0$

$$z_r = \cos\left(\frac{0}{n} + \frac{2\pi r}{n}\right) + isin\left(\frac{0}{n} + \frac{2\pi r}{n}\right)$$

And if $R = -r,\ z_R = \cos\left(\frac{0}{n} - \frac{2\pi r}{n}\right) + isin\left(\frac{0}{n} - \frac{2\pi r}{n}\right) = {z_r}^*$

Example 2: $z^n = -1 = cos\pi, +isin\pi$

$$z_r = \cos\left(\frac{\pi}{n} + \frac{2\pi r}{n}\right) + isin\left(\frac{\pi}{n} + \frac{2\pi r}{n}\right)$$

And if $R = -1 - r,\ z_R = \cos\left(\frac{\pi}{n} - \frac{2\pi}{n} - \frac{2\pi r}{n}\right) + isin\left(\frac{\pi}{n} - \frac{2\pi}{n} - \frac{2\pi r}{n}\right)$

$$= \cos\left(-\frac{\pi}{n} - \frac{2\pi r}{n}\right)$$

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Solution

z can be written as $x - xi$, where $x > 0$,

so that $(x - xi)(1 + ai) = a + i$

and $x + xai - xi + xa = a + i$

Then equating real and imaginary parts:

$x + xa = a$ & $xa - x = 1$;

ie $x(1 + a) = a$ & $x(a - 1) = 1$,

so that $x = \frac{a}{1+a} = \frac{1}{a-1}$

and $a^2 - a = 1 + a$

$\Rightarrow a^2 - 2a - 1 = 0$

$\Rightarrow a = \frac{2\pm\sqrt{8}}{2} = 1 \pm \sqrt{2}$

Also $x > 0$:

$a = 1 \pm \sqrt{2} \Rightarrow x = \frac{1}{a-1} = \frac{1}{\pm\sqrt{2}}$

so that $a = 1 + \sqrt{2}$

Find the modulus and argument of $e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}}$

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Solution

Method 1

Write $z = e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}}$ in the form $e^{a\pi i}(e^{b\pi i} - e^{-b\pi i})$

So $a + b = \frac{7}{10}$ & $a - b = -\frac{9}{10}$

Then $a = -\frac{1}{10}$ & $b = \frac{8}{10}$

and $e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}} = e^{-\frac{\pi i}{10}}(e^{\frac{8\pi i}{10}} - e^{-\frac{8\pi i}{10}})$

$= e^{-\frac{\pi i}{10}}(2isin\left(\frac{4\pi}{5}\right))$

Then $|z| = \left|e^{-\frac{\pi i}{10}}\right| \left|2isin\left(\frac{4\pi}{5}\right)\right| = (1)(2sin\left(\frac{4\pi}{5}\right))$

$= 2\sin\left(\pi - \frac{4\pi}{5}\right) = 2\sin\left(\frac{\pi}{5}\right)$

and $\arg(z) = \arg\left(e^{-\frac{\pi i}{10}}\right) + \arg\left(2isin\left(\frac{4\pi}{5}\right)\right)$

$= -\frac{\pi}{10} + \frac{\pi}{2} = \frac{4\pi}{10} = \frac{2\pi}{5}$

Method 2

$e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}}$

$= \left(cos\left(\frac{7\pi}{10}\right) - cos\left(\frac{-9\pi}{10}\right)\right) + i\left(sin\left(\frac{7\pi}{10}\right) - sin\left(\frac{-9\pi}{10}\right)\right)$

$= -2sin\left(\frac{1}{2}\left(\frac{7\pi}{10} + \frac{-9\pi}{10}\right)\right) sin\left(\frac{1}{2}\left(\frac{7\pi}{10} - \frac{-9\pi}{10}\right)\right)$

$+2cos\left(\frac{1}{2}\left(\frac{7\pi}{10} + \frac{-9\pi}{10}\right)\right) sin\left(\frac{1}{2}\left(\frac{7\pi}{10} - \frac{-9\pi}{10}\right)\right)$

$$= -2sin\left(-\frac{\pi}{10}\right)sin\left(\frac{8\pi}{10}\right) + 2icos\left(-\frac{\pi}{10}\right)sin\left(\frac{8\pi}{10}\right)$$

$$= 2sin\left(\frac{8\pi}{10}\right)\left\{sin\left(\frac{\pi}{10}\right) + icos\left(\frac{\pi}{10}\right)\right\}$$

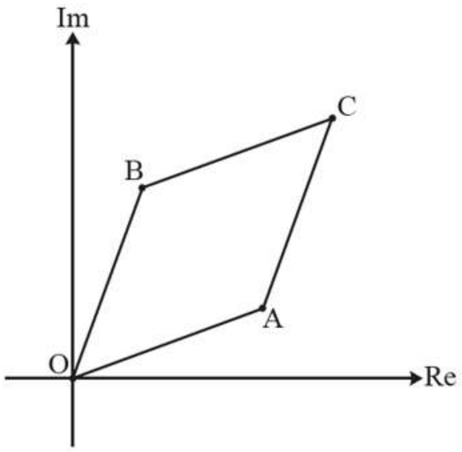
$$= 2sin\left(\frac{4\pi}{5}\right)\left\{cos\left(\frac{\pi}{2} - \frac{\pi}{10}\right) + isin\left(\frac{\pi}{2} - \frac{\pi}{10}\right)\right\}$$

$$= 2sin\left(\frac{\pi}{5}\right)\left\{cos\left(\frac{4\pi}{10}\right) + isin\left(\frac{4\pi}{10}\right)\right\}$$

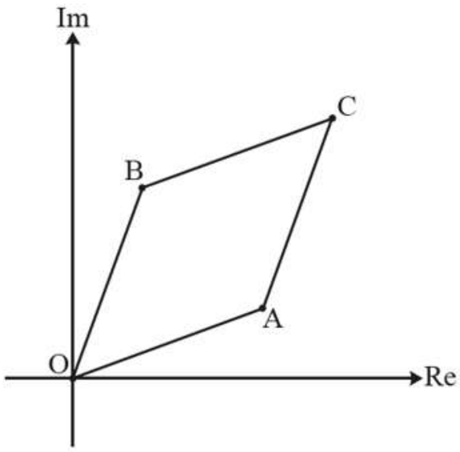
$$= 2sin\left(\frac{\pi}{5}\right)e^{\frac{2\pi i}{5}}$$

So mod is $2sin\left(\frac{\pi}{5}\right)$ and arg is $\frac{2\pi}{5}$

Referring to the diagram, use complex numbers to prove that the diagonal OC of the rhombus OACB bisects the angle OAB.



[Referring to the diagram, use complex numbers to prove that the diagonal OC of the rhombus OACB bisects the angle OAB.]



Solution

Let z & w be the complex numbers represented by the points A & B. Write $z + w = re^{i\theta} + re^{i(\theta+\alpha)}$, where $\alpha = \angle AOB$

[aiming to show that arg $(z + w)$ will be $\theta + \frac{\alpha}{2}$]

Then $z + w = re^{i\left(\theta+\frac{\alpha}{2}\right)}(e^{-i\frac{\alpha}{2}} + e^{i\frac{\alpha}{2}})$

$= re^{i\left(\theta+\frac{\alpha}{2}\right)}.\ 2\cos(\frac{\alpha}{2})$,

and hence $\arg(z + w) = \theta + \frac{\alpha}{2} = \frac{1}{2}(\theta + [\theta + \alpha])$

$= \frac{1}{2}(argz + argw)$

Then, as C represents $z + w$, OC bisects the angle OAB.