Collisions - Conditions for reversal of direction

(6 pages; 24/6/20)

Particles A and B, of masses km & m, respectively (where $k > 0$), are travelling on the same straight path, on a smooth surface, and collide. The coefficient of restitution between A and B is e. Investigate the conditions that must apply in order for A to change direction, in the following situations:

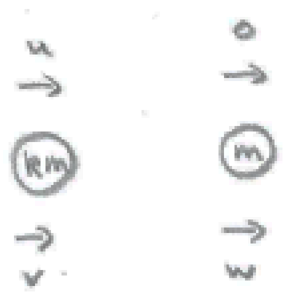
(I) A has speed u and B is stationary

(II) A and B are moving in the same direction; A has speed λu ($\lambda > 1$) and B has speed u.

(III) A and B are moving in opposite directions; A has speed θu and B has speed u.

Solution

(I)



$CoM \Rightarrow ku = kv + w$ (1) & $NLI \Rightarrow w - v = eu$ (2)

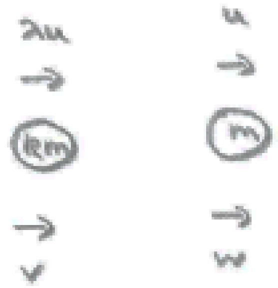
$(1) - (2) \Rightarrow (k+1)v = (k-e)u$

$\Rightarrow v = \frac{(k-e)u}{k+1}$; so $v < 0 \Leftrightarrow e > k$

Thus if $k \geq 1$, a change of direction isn't possible.

If $k < 1$, a change of direction will be possible provided e is sufficiently big. Note that a bigger e means that A and B bounce off each other more.

(II)



$CoM \Rightarrow \lambda ku + u = kv + w$ (1) & $NLI \Rightarrow w - v = e(\lambda - 1)u$ (2)

$(1) - (2) \Rightarrow (k+1)v = u(\lambda k + 1 - e\lambda + e)$

$\Rightarrow v = \frac{u(\lambda k + 1 - e\lambda + e)}{k+1}$

So $v < 0 \Leftrightarrow \lambda k + 1 - e\lambda + e < 0$

$\Leftrightarrow \lambda k + 1 < e(\lambda - 1)$

$\Leftrightarrow e > \frac{\lambda k + 1}{\lambda - 1} \quad (\lambda > 1, k > 0)$

Conclusions

(a) A will never reverse direction if $\frac{\lambda k + 1}{\lambda - 1} \geq 1$ (as e cannot exceed 1) $\Leftrightarrow \lambda k + 1 \geq \lambda - 1$ (as $\lambda > 1$) $\Leftrightarrow \lambda(1 - k) \leq 2$

So A will never reverse direction if $k \geq 1$ or $\lambda \leq 2$ (as $k > 0$, so that $1 - k < 1$).

(b) Suppose that the momentum of A equals that of B, so that $\lambda k = 1$. Then $\frac{\lambda k + 1}{\lambda - 1} = 1 \Leftrightarrow \frac{2}{\lambda - 1} = 1 \Leftrightarrow 2 = \lambda - 1$; ie $\lambda = 3$

So a critical point occurs when the momentums are equal and

$\lambda = 3$. The following table can be produced (the results are derived later):

Reversal of direction of A

	$\lambda k < 1$	$\lambda k = 1$	$\lambda k > 1$
$\lambda \le 2$	X	X	X
$2 < \lambda < 3$	Y*	X	X
$\lambda = 3$	YY	X	X
$\lambda > 3$	YY	YY	Y*

X: never

Y: for big enough e, but with further constraints on λ & k

YY: for big enough e

$* \; k < \frac{\lambda-2}{\lambda}$

[And for reversal to occur, $0 < k < 1$, in all cases.]

Observations from table

(i) When the momentum of A is less than that of B:

A will reverse direction if $\lambda \geq 3$ (ie if the speed of A is sufficiently high) and e is big enough.

If $\lambda < 3$, A will reverse direction for suitable λ & k (provided e is big enough).

(ii) When A and B have the same momentum:

A will reverse direction if $\lambda > 3$ (ie if the speed of A is sufficiently high) and e is big enough.

If $\lambda \leq 3$, A will never reverse direction.

(iii) When the momentum of A is greater than that of B:

If $\lambda > 3$, A will reverse direction for suitable λ & k (provided e is big enough).

If $\lambda \leq 3$, A will never reverse direction.

Derivation of results in the table

($\lambda > 1, k > 0$ throughout)

(A) When $\lambda k = 1, \frac{\lambda k+1}{\lambda-1} = \frac{2}{\lambda-1}$

Then reversal is possible when $\frac{2}{\lambda-1} < 1 \Leftrightarrow 2 < \lambda - 1$ (as $\lambda > 1$); ie $\lambda > 3$ (this enables the $\lambda k = 1$ column of the table to be completed).

(B) Suppose that $\lambda k < 1$

$\frac{\lambda k+1}{\lambda-1} < 1 \Leftrightarrow \lambda k + 1 < \lambda - 1 \Leftrightarrow \lambda k < \lambda - 2$ (*)

When $\lambda \geq 3$, RHS of (*) ≥ 1 and LHS < 1, so that (*) is always satisfied (and reversal occurs for sufficiently big e).

When $\lambda < 3$: (*) $\Leftrightarrow k < \frac{\lambda-2}{\lambda}$ (eg if $\lambda = 2.5, k < 0.2$)

ie reversal occurs (for sufficiently big e) if k is sufficiently small.

(C) Suppose that $\lambda k > 1$

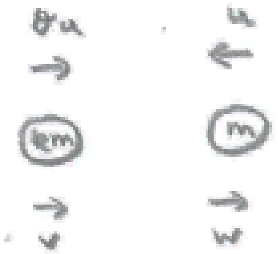
Then, as before, $\frac{\lambda k+1}{\lambda-1} < 1 \Leftrightarrow \lambda k < \lambda - 2$ (*)

When $\lambda \leq 3$, RHS of (*) ≤ 1 and LHS > 1, so that (*) is never satisfied.

When $\lambda > 3$: (*) $\Leftrightarrow k < \frac{\lambda-2}{\lambda}$ (eg if $\lambda = 4, k < 0.5$)

ie reversal occurs (for sufficiently big e) if k is sufficiently small.

(III)



($\theta > 1$ [ie, without loss of generality, the faster moving particle is assumed to arrive from the left] and $k > 0$)

$CoM \Rightarrow \theta ku - u = kv + w$ (1) & $NLI \Rightarrow w - v = e(\theta + 1)u$ (2)

$(1) - (2) \Rightarrow (k+1)v = u(\theta k - 1 - e\theta - e)$

$\Rightarrow v = \frac{u(\theta k - 1 - e\theta - e)}{k+1}$

So $v < 0 \Leftrightarrow \theta k - 1 - e\theta - e < 0$

$\Leftrightarrow \theta k - 1 < e(\theta + 1)$

$\Leftrightarrow e > \frac{\theta k - 1}{\theta + 1}$

Conclusions

A will reverse direction if $\frac{\theta k - 1}{\theta + 1} < 1$ (for sufficiently big e)

$\Leftrightarrow \theta k - 1 < \theta + 1 \Leftrightarrow \theta(k-1) < 2$

If $k \leq 1$, then A will reverse direction, for sufficiently big e.

If $\boldsymbol{k} \geq 3$, then A will never reverse direction (as $\theta > 1$).

If $1 < k < 3, \theta(k-1) < 2 \Leftrightarrow k < \frac{2}{\theta} + 1 = \frac{\theta+2}{\theta}$

The condition $\boldsymbol{k} < \frac{\boldsymbol{\theta}+2}{\boldsymbol{\theta}}$ (and sufficiently big $\boldsymbol{e}$) is in fact necessary and sufficient for A to reverse direction.