Centre of Mass - Exercises (Solutions)

(9 pages; 25/3/20)

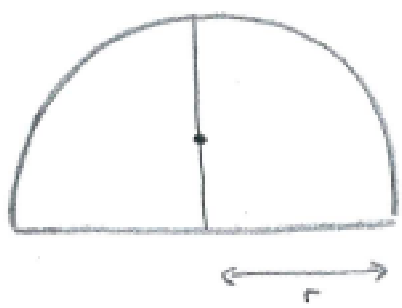
(1*) Centre of Mass of Lamina by Integration

Find the centre of mass of a semi-circular lamina of radius r.

(a) by integrating wrt x

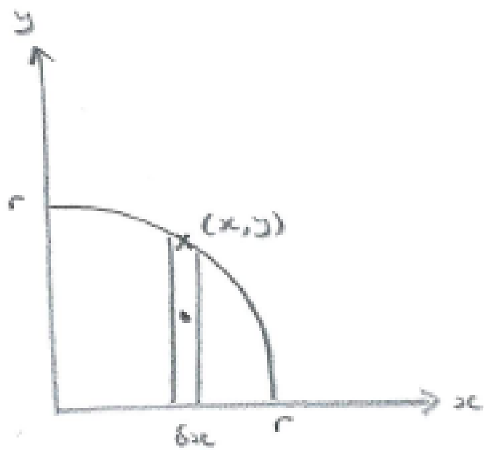
(b) by integrating wrt y

Solution



By symmetry, the CoM lies halfway across, as shown in the diagram. We just need to find the y component, $\bar{y}$. Again, by symmetry, $\bar{y}$ for a quarter circle will equal $\bar{y}$ for the semi-circle.

(a)



Considering vertical strips as shown in the diagram,

each strip has area $y\delta x$, and the total area of the quarter circle is $\frac{1}{4}\pi r^2$. $\bar{y}$ for each strip is $\frac{y}{2}$.

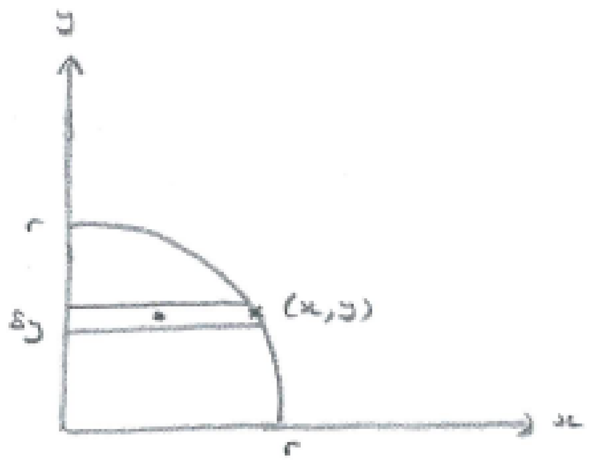
Weighting the CoM of each strip by its area and summing gives

$$\bar{y} = \lim_{\delta x \to 0} \frac{1}{\left(\frac{\pi r^2}{4}\right)} \sum \frac{y}{2} . y\delta x$$

$$= \frac{1}{\left(\frac{\pi r^2}{4}\right)} \int_0^r \frac{y}{2} . ydx = \frac{2}{\pi r^2} \int_0^r r^2 - x^2 dx, \quad \text{since } x^2 + y^2 = r^2$$

$$= \frac{2}{\pi r^2} \left[r^2 x - \frac{1}{3} x^3 \right]_0^r = \frac{2}{\pi r^2} \left(\frac{2r^3}{3} \right) = \frac{4r}{3\pi}$$

(b)



Considering horizontal strips as shown in the diagram,

each strip has area $x\delta y$, and $\bar{y}$ for each strip is y.

Then $\bar{y} = \frac{1}{\left(\frac{\pi r^2}{4}\right)} \int_0^r y . xdx = \frac{4}{\pi r^2} \int_0^r \sqrt{r^2 - x^2} . xdx$

As $\frac{d}{dx}(r^2 - x^2) = -2x$ and we have an x in the integrand,

we can make the substitution $u = r^2 - x^2$, so that $du = -2xdx$

and $\overline{y} = \frac{4}{\pi r^2}\int_{r^2}^{0} \sqrt{u}.\left(-\frac{1}{2}\right) du = \frac{2}{\pi r^2}\int_{0}^{r^2} \sqrt{u}.\, du$

$= \frac{2}{\pi r^2}\left[\frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^{r^2} = \frac{4r}{3\pi}$, as before

(2**) Centre of mass of solid of revolution

The region between the curve $y = x^3 - x^2$ and the x-axis is rotated by 360° about the x-axis. Find the centre of mass of the solid of revolution obtained.

Solution

By symmetry, $\bar{y} = 0$

$y = x^3 - x^2 = x^2(x-1)$ meets the x-axis at $x = 0$ & $x = 1$

$V\bar{x} = \int_0^1 x(\pi y^2 dx)$, where $V = \int_0^1 \pi y^2 dx$

[$\pi y^2 dx$ is the volume of the disc of width dx at position x]

Thus $V = \pi\int_0^1 x^4(x-1)^2 dx = \pi\int_0^1 x^6 - 2x^5 + x^4\, dx$

$= \pi[\frac{1}{7}x^7 - \frac{2}{6}x^6 + \frac{1}{5}x^5]_0^1$

$= \pi\left(\frac{1}{7} - \frac{2}{6} + \frac{1}{5}\right) = \frac{(30-70+42)\pi}{210} = \frac{\pi}{105}\ units^3$

And $\bar{x} = 105\int_0^1 x^7 - 2x^6 + x^5\, dx$

$= 105[\frac{1}{8}x^8 - \frac{2}{7}x^7 + \frac{1}{6}x^6]_0^1$

$= 105\left(\frac{1}{8} - \frac{2}{7} + \frac{1}{6}\right) = \frac{105(21-48+28)}{168}$

$= \frac{105}{168} = \frac{35}{56} = \frac{5}{8} = 0.625$

Thus the centre of mass is $(0.625, 0)$

(3**) Centre of mass of lamina

Find the centre of mass of the region between the curve

$y = x^3 - x^2$ and the x-axis.

Solution

$y = x^3 - x^2 = x^2(x-1)$ meets the x-axis at $x = 0$ & $x = 1$

Note that the curve lies beneath the x-axis.

Total weight (signed area)

$= \int_0^1 x^3 - x^2\, dx = \left[\frac{1}{4}x^4 - \frac{1}{3}x^3\right]_0^1 = \left(\frac{1}{4} - \frac{1}{3}\right) = -\frac{1}{12}$

$-\frac{1}{12}\bar{x} = \int_0^1 x(x^3 - x^2\, dx) = \left[\frac{1}{5}x^5 - \frac{1}{4}x^4\right]_0^1 = \left(\frac{1}{5} - \frac{1}{4}\right) = -\frac{1}{20}$

so that $\bar{x} = 0.6$

And $-\frac{1}{12}\bar{y} = \int_0^1 \frac{y}{2}(x^3 - x^2\, dx) = \frac{1}{2}\int_0^1 (x^3 - x^2)^2 dx$

$= \frac{1}{2}\int_0^1 x^6 + x^4 - 2x^5\, dx$

$= \frac{1}{2}[\frac{1}{7}x^7 + \frac{1}{5}x^5 - \frac{2}{6}x^6]_0^1$

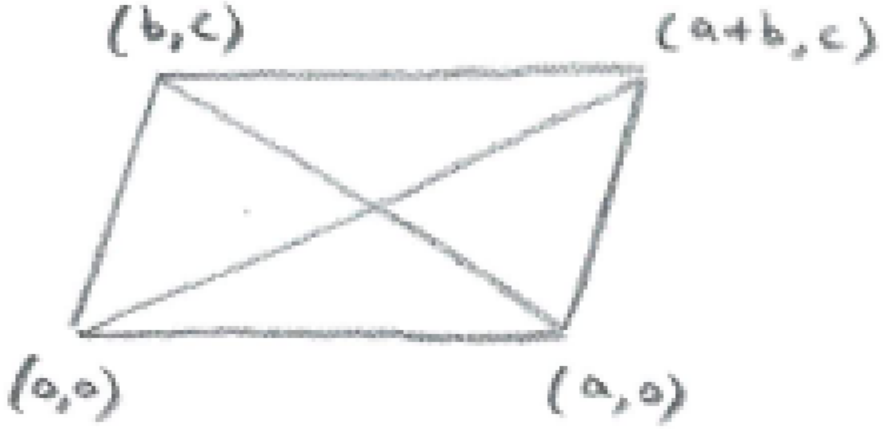
$= \frac{1}{2}\left(\frac{1}{7} + \frac{1}{5} - \frac{2}{6}\right)$

so that $\bar{y} = \frac{-6(30+42-70)}{210} = -\frac{2}{35} = -0.0571$ (3sf)

Thus the centre of mass is $(0.6, -0.0571)$

(4***) Show that the centre of mass of a parallelogram is at the intersection of the diagonals, by finding the centre of mass of two triangles, given the result that the diagonals bisect each other.

Solution



Let triangle 1 have corners

(0,0) , (a,0) & (b,c)

(and triangle 2 be the other half of the parallelogram).

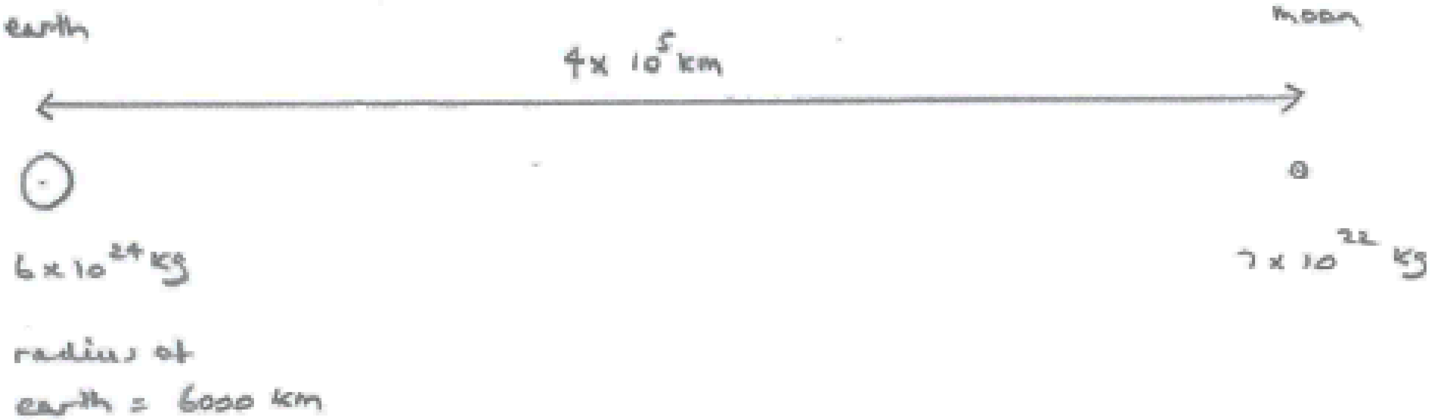
$$CoM_1 = \begin{pmatrix} \frac{1}{3}(0+a+b) \\ \frac{1}{3}(0+0+c) \end{pmatrix} \ \& \ CoM_2 = \begin{pmatrix} \frac{1}{3}(a+b+[a+b]) \\ \frac{1}{3}(0+c+c) \end{pmatrix}$$

Then centre of mass of parallelogram $= \frac{1}{2}\ (COM_1 + COM_2)$

$$= \frac{1}{6}\begin{pmatrix} 3a+3b \\ 3c \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a+b \\ c \end{pmatrix}$$

ie the mid-point of the diagonal from (0,0) to (a+b,c)

(5**) Find the centre of mass of the Earth-Moon system



distance from earth to moon = 4x10^5km

radius of earth = 6000km

mass of earth = 6x10^{24}kg

mass of moon = 7x10^{22}kg

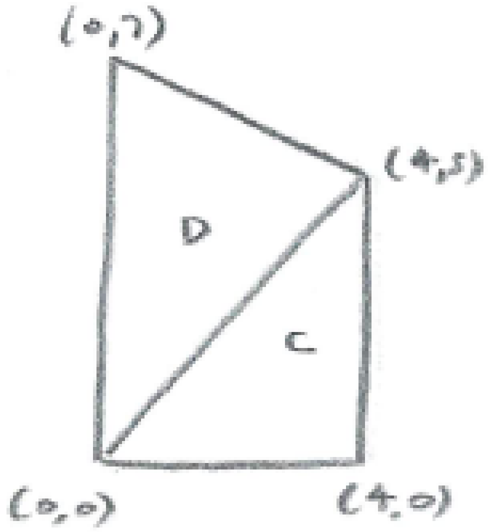
Solution

Taking the centre of the earth as the origin,

$(6\times10^{24}+7\times10^{22})\,\bar{x} = 6\times10^{24}\times0\ +7\times10^{22}\times4\times10^5$

$\Rightarrow\ \bar{x} = \frac{28\times10^{27}}{607\times10^{22}} = 4600$ km (2sf)

(6**) Find the centre of mass of the trapezium in the diagram, by dividing it up as shown.



Solution

CoM of C: $\begin{pmatrix} \frac{1}{3}(0+4+4) \\ \frac{1}{3}\ (0+0+5) \end{pmatrix} = \begin{pmatrix} \frac{8}{3} \\ \frac{5}{3} \end{pmatrix}$

CoM of D: $\begin{pmatrix} \frac{1}{3}(0+4+0) \\ \frac{1}{3}\ (0+5+7) \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ 4 \end{pmatrix}$

Area of trapezium $= \frac{1}{2}(5+7)(4) = 24$

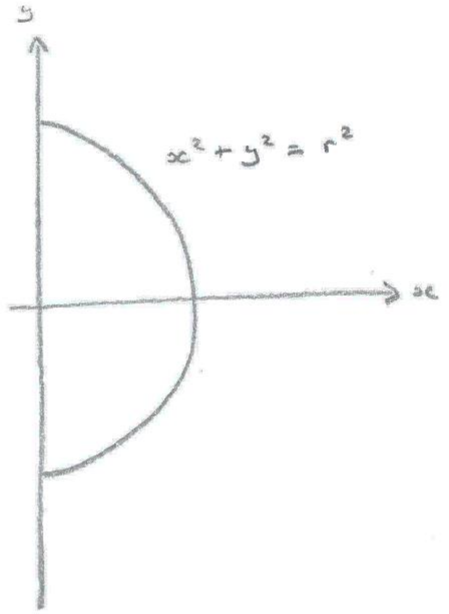
Area of C $= \frac{1}{2}(4)(5) = 10$

$\therefore$ Area of D $= 14$

$$\text{CoM of trapezium} = \frac{1}{24}\begin{pmatrix} 10\left(\frac{8}{3}\right) + 14\left(\frac{4}{3}\right) \\ 10\left(\frac{5}{3}\right) + 14(4) \end{pmatrix}$$

$$= \frac{1}{24}\begin{pmatrix} \frac{136}{3} \\ \frac{218}{3} \end{pmatrix} = \begin{pmatrix} \frac{17}{9} \\ \frac{109}{36} \end{pmatrix}$$

(7**) Find the centre of mass of the semi-circular lamina shown in the diagram.



Solution

By symmetry, we need only consider the top half :

$$\bar{x} = \frac{1}{\frac{1}{4}\pi r^2}\int_0^r xy\,dx = \frac{4}{\pi r^2}\int_0^r x\sqrt{r^2 - x^2}\,dx$$

$$= \frac{-2}{\pi r^2}\int_0^r (-2x)\sqrt{r^2 - x^2}\,dx$$

$$= \frac{-2}{\pi r^2}\left[\frac{(r^2 - x^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^r$$

$$= -\frac{4}{3\pi r^2}(0 - r^3)$$

$$= \frac{4r}{3\pi}$$