Circular Motion - Constant Speed (9 pages; 25/10/20)

[Vertical circular motion, where the speed is variable, is covered in a separate note. See also the note on "Fictitious Forces", which covers 'centrifugal' forces.]

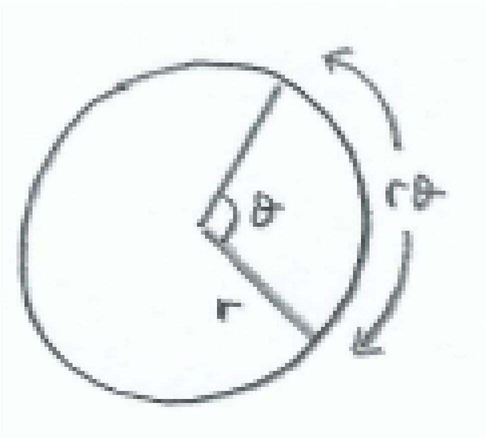
(1) Angular Speed

arc length $s = r\theta$

Let $\omega = \dot{\theta} = \frac{d\theta}{dt}$

Then $v = \frac{ds}{dt} = \frac{d}{dt}(r\theta) = r\frac{d\theta}{dt} = r\omega$

Constant $\omega \Rightarrow \omega = \frac{\theta}{t}$ or $\theta = \omega t$



(2) Example: Coin on a record turntable

Turntable rotates at 45 revolutions per minute (rpm)

To calculate the angular speed ω in rads^{-1}:

$\omega = \frac{45(2\pi)}{60} = 4.7124 = 4.71$ rads^{-1}(3sf)

(3) Example

Find the speed of a coin that is 8cm from the centre of a record on the turntable in (2) (assuming that it doesn't move relative to the surface of the record).

Solution

$v = r\omega = 0.08(4.7124) = 0.37699 = 0.377\ ms^{-1}$ (3sf)

(4) Example: car wheel

Suppose $r = 0.25m$

Car travels at $60\ kmph$

Arc length covered in 3600 s is 60 000 m

$\Rightarrow v = \frac{60000}{3600} = \frac{100}{6} = \frac{50}{3}$

$\Rightarrow \omega = \frac{v}{r} = \frac{50}{3(0.25)} = 67\ rads^{-1}$

(5) Example: Earth's orbit about the Sun

r = 1.5 x 10^{11}m (150 million km)

Find

(a) speed of earth (v)

(b) angular speed (ω)

Solution

$v = \frac{2\pi(1.5)(10^{11})}{365(24)(3600)} \approx 30000\ ms^{-1}$ (30 kms^{-1})

$\Rightarrow \omega \approx \frac{30000}{1.5\times10^{11}} = 2 \times 10^{-7} rads^{-1}$

Or $\omega = \frac{2\pi}{365\times24\times3600}\ rads^{-1}$

(6) Velocity vector

$\boldsymbol{r} = rcos\theta\ \boldsymbol{i} + rsin\theta\ \boldsymbol{j} = rcos\omega t\ \boldsymbol{i} + rsin\omega t\ \boldsymbol{j}$

$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = -\omega rsin\omega t\ \boldsymbol{i} + \omega rcos\omega t\ \boldsymbol{j} = -\omega rsin\theta\ \boldsymbol{i} + \omega rcos\theta\ \boldsymbol{j}$

$v = |\boldsymbol{v}| = \sqrt{\omega^2 r^2(sin^2\theta + cos^2\theta)} = \omega r$

Exercise: Show that the direction of **v** is perpendicular to $\boldsymbol{r}$.

(7) Acceleration vector

$$\boldsymbol{a} = \frac{dv}{dt} = -\omega^2 r cos\omega t\ \boldsymbol{i}\ - \omega^2 r sin\omega t\ \boldsymbol{j}$$

$$|\boldsymbol{a}| = \omega^2 r \quad \text{or} \quad \left(\frac{v}{r}\right)^2 r = \frac{v^2}{r}$$

Direction of $\boldsymbol{a}$ is opposite that of $\boldsymbol{r}$.

The centripetal acceleration is thus towards the centre of the circle.

(8) Centripetal Force

$$F = ma = \frac{mv^2}{r} \quad (\text{or } m\omega^2 r)$$

The centripetal force (towards the centre of the circle) is what causes the object to deviate from a straight line path.

The centripetal force is sometimes referred to as a resultant force. This only means that there is a net force on the object towards the centre (though there might only be one source of force; ie not strictly a resultant force).

Note that, in most Mechanics problems, the forces are known and the acceleration is then determined. In the case of circular motion, however, the acceleration is deduced from the fact that the motion is known to be circular, and the centripetal force is then determined from the acceleration.

See "Fictitious Forces" for a discussion of 'centrifugal' force, and its relation to centripetal force.

(9) Sources of the centripetal force

We will meet the following sources later on:

- tension in a string or rod

- reaction of a surface on the object

- a component of the object's weight (for vertical circular motion, covered in a separate note)

- friction

- gravitational force from the centre (in the case of satellites etc)

(10) Example

If the coin in the earlier example has a mass of 7g, find the frictional force needed to maintain it in its position.

$[\omega = 4.7124\ ; r = 0.08; v = 0.37699\ ms^{-1}]$

Solution

N2L $\Rightarrow \quad F = (0.007)(0.08)(4.7124)^2 = 0.012436 = 0.0124N$

(11) Example

A conveyor belt in a sweet factory carries a chocolate of mass 30g round a bend of radius 50cm. If the coefficient of friction between the chocolate and the surface is 0.2, what is the greatest angular speed with which the chocolate can be carried round the bend?

Solution

$F = ma\ ; \quad F = \mu R\ ; \quad R = mg\ ; \ a = \omega^2 r$

$\Rightarrow \mu mg = m\omega^2 r$

$\Rightarrow \omega^2 = \frac{\mu g}{r}$

$$\Rightarrow \omega = \sqrt{\frac{0.2\times 9.8}{0.5}} = 1.98 \text{ rad}s^{-1}$$

Note that the mass cancels.

(12) Example: Earth's orbit (again)

Newton's law of gravitation: $F = \frac{GMm}{r^2}$

where M = mass of Sun ≈ 2 x 10^{30} kg, m = mass of earth

r = 1.5 x 10^{11}m, $G \approx 7 \text{ x } 10^{-11}$

So $\frac{GMm}{r^2} = \frac{mv^2}{r}$

(13) Example

Determine v from the above relation.

Solution

M ≈ 2 x 10^{30} kg , r = 1.5 x 10^{11}m, $G \approx 7 \text{ x } 10^{-11}$

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow v^2 = \frac{GM}{r} \approx \frac{7\times 10^{-11}\times 2\times 10^{30}}{1.5\times 10^{11}} = \frac{28}{3}\times 10^8$$

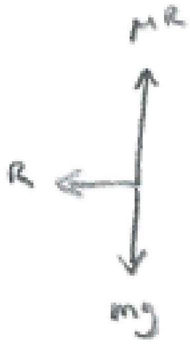
$$\Rightarrow v = 3.06\times 10^4 ms^{-1} \quad (\approx 30\ kms^{-1})$$

(14) Example

At a fairground 'wall of death', participants are invited to stand against a wall inside a cylinder of radius 5m, which then rotates. Once a suitable speed has been attained, the floor is removed. If the coefficient of friction between the wall and the participant is 0.2, with what angular speed must the cylinder be rotating in order for the participant not to slip down the wall? How fast is the individual moving?

Solution

A force diagram can be constructed for the participant, in the limiting case where he or she is about to slide down the wall - where R is the reaction of the wall, m is the mass of the participant and μ is the coefficient of friction.



Vertical equilibrium $\Rightarrow \mu R = mg$ (1)

Circular motion $\Rightarrow R = m\omega^2(5)$ (2)

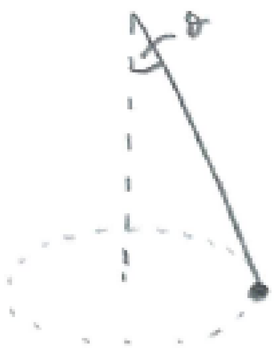
Then (1) & (2) $\Rightarrow \frac{mg}{\mu} = 5m\omega^2$

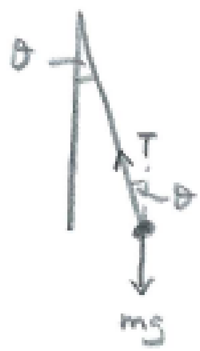
and hence $\omega = \sqrt{\frac{g}{5\mu}} = \sqrt{9.81} =$
$3.13\ rads^{-1}(3sf)$

[Reasonableness check: this is almost exactly 1 revolution every 2 seconds.]

And $v = \omega r = \sqrt{9.81}(5) = 15.660 = 15.7\ ms^{-1}\ (3sf)$

(15) Conical Pendulum





Resolving vertically, equilibrium $\Rightarrow Tcos\theta = mg$

Resolving horizontally, circular motion $\Rightarrow Tsin\theta = \frac{mv^2}{r} = \frac{mv^2}{lsin\theta}$

if l is the length of the string

(16) Example

Find ω as a function of $cos\theta$

Solution

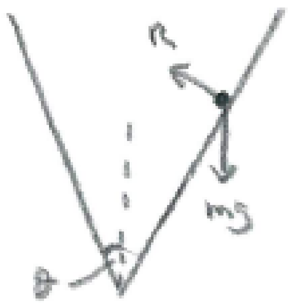
$Tcos\theta = mg$, $\quad Tsin\theta = \frac{mv^2}{lsin\theta}$, $v = r\omega$, $r = lsin\theta$

$\Rightarrow (r\omega)^2 = \frac{Tlsin^2\theta}{m} = \frac{mglsin^2\theta}{mcos\theta} = \frac{gl^2sin^2\theta}{lcos\theta} = \frac{gr^2}{lcos\theta}$

$\Rightarrow \omega^2 = \frac{g}{lcos\theta}$

(17) Example

Set up equations for the inverted cone shown below, by resolving horizontally and vertically, if the particle is at height h.



Solution

Vertically: $Rsin\theta = mg$

Horizontally: $Rcos\theta = \frac{mv^2}{r} = \frac{mv^2}{htan\theta}$

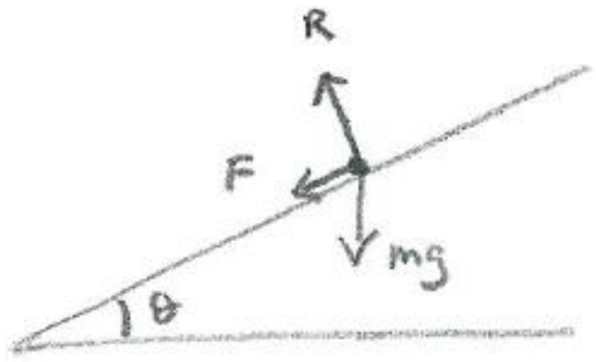
(18) Banked Track

Consider a car travelling round a circular bend on a banked track.

Newton's 1st Law says that the car will attempt to continue in straight line motion. There is also gravity and possibly friction acting on the car.

At low speeds, gravity is the dominant effect, and the tendency is for the car to slip down the bank. As friction opposes motion, it therefore acts up the bank in this case.

At high speeds, Newton's 1st Law is the dominant effect, and the tendency is for the car to slip up the bank. In this case friction acts down the bank (as shown in the diagram below).



At some intermediate speed, there is no tendency to slip either up or down the bank, and so there is no frictional force. Some exam questions will state that this situation applies.

If friction acts down the slope:

Vertical equilibrium ⇒

$Rcos\theta = Fsin\theta + mg$

$\Rightarrow R(cos\theta - \mu sin\theta) = mg \quad (1)$

(Horizontal) circular motion $\Rightarrow$

$$Fcos\theta + Rsin\theta = \frac{mv^2}{r}$$

$$\Rightarrow R(\mu cos\theta + sin\theta) = \frac{mv^2}{r} \quad (2)$$

$$(1)\&(2) \Rightarrow R = \frac{mg}{cos\theta - \mu sin\theta} = \frac{mv^2}{r(\mu cos\theta + sin\theta)}$$

$$\Rightarrow v^2 = \frac{gr(\mu cos\theta + sin\theta)}{cos\theta - \mu sin\theta}$$

Note that the (horizontal) centripetal force depends on gravity (which is vertical). This may seem paradoxical until you consider that the limiting frictional force for a block on a horizontal surface is μR = μmg.