Centre of Mass – Q1 [12 marks](1/6/21)

Exam Boards

OCR : Mechanics (Year 2)

MEI: Mechanics b

AQA: Mechanics (Year 2)

Edx: Mechanics 2 (Year 2)

Find the centre of mass of a semi-circular lamina of radius r.

(a) by integrating wrt x [7 marks]

(b) by integrating wrt y [5 marks]

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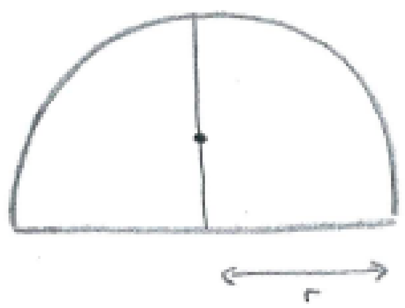
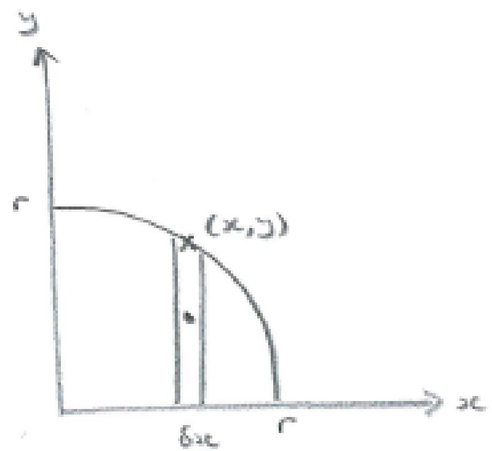
Solution



By symmetry, the CoM lies halfway across, as shown in the diagram. [1 mark]

We just need to find the y component, $\overline{y}$. Again, by symmetry, $\overline{y}$ for a quarter circle will equal $\overline{y}$ for the semi-circle. [1 mark]

(a)



Considering vertical strips as shown in the diagram, each strip has area $y\delta x$, and the total area of the quarter circle is $\frac{1}{4}\pi r^2$. $\overline{y}$ for each strip is $\frac{y}{2}$.

Weighting the CoM of each strip by its area and summing gives

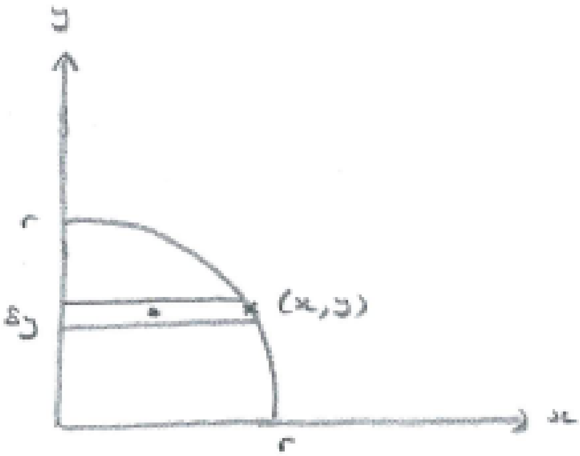
$$\overline{y} = \lim_{\delta x \to 0} \frac{1}{\left(\frac{\pi r^2}{4}\right)} \sum \frac{y}{2} . y\delta x$$

$= \frac{1}{\left(\frac{\pi r^2}{4}\right)} \int_0^r \frac{y}{2} . ydx$ [2 marks]

$= \frac{2}{\pi r^2} \int_0^r r^2 - x^2 dx$, since $x^2 + y^2 = r^2$ [1 mark]

$= \frac{2}{\pi r^2} \left[r^2 x - \frac{1}{3} x^3 \right]_0^r = \frac{2}{\pi r^2} \left(\frac{2r^3}{3} \right) = \frac{4r}{3\pi}$ [2 marks]

(b)



Considering horizontal strips as shown in the diagram,

each strip has area $x\delta y$, and $\overline{y}$ for each strip is y.

Then $\overline{y} = \frac{1}{\left(\frac{\pi r^2}{4}\right)} \int_0^r y . xdx$

$= \frac{4}{\pi r^2} \int_0^r \sqrt{r^2 - x^2} . xdx$ [2 marks]

As $\frac{d}{dx}(r^2 - x^2) = -2x$ and we have an x in the integrand,

we can make the substitution $u = r^2 - x^2$, so that $du = -2xdx$

and $\overline{y} = \frac{4}{\pi r^2}\int_{r^2}^{0} \sqrt{u}.\left(-\frac{1}{2}\right) du = \frac{2}{\pi r^2}\int_{0}^{r^2} \sqrt{u}.du$ [2 marks]

$= \frac{2}{\pi r^2}\left[\frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^{r^2} = \frac{4r}{3\pi}$, as before [1 mark]