Centre of Mass - Exercises (with Sol'ns)

(15 pages; 7/2/26)

(1) Find the centre of mass of a semi-circular lamina of radius r.

(a) by integrating wrt x

(b) by integrating wrt y

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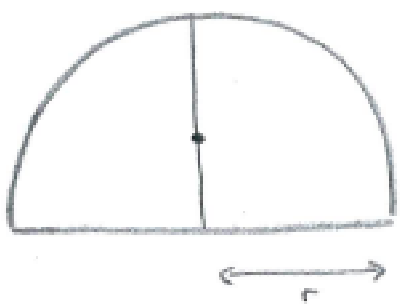
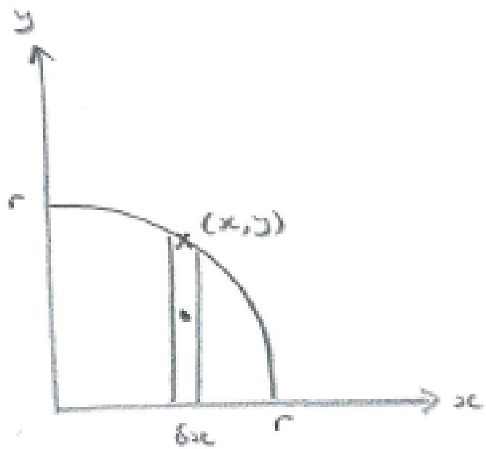
Solution



By symmetry, the CoM lies halfway across, as shown in the diagram.

We just need to find the y component, $\bar{y}$. Again, by symmetry, $\bar{y}$ for a quarter circle will equal $\bar{y}$ for the semi-circle.

(a)



Considering vertical strips as shown in the diagram, each strip has area $y\delta x$, and the total area of the quarter circle is $\frac{1}{4}\pi r^2$. $\overline{y}$ for each strip is $\frac{y}{2}$.

Weighting the CoM of each strip by its area and summing gives

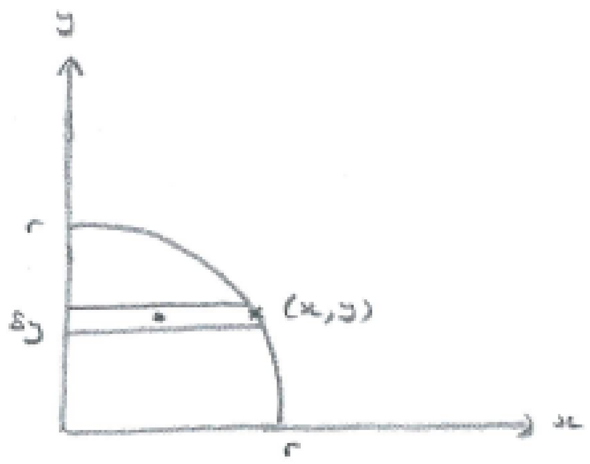
$$\overline{y} = \lim_{\delta x \to 0} \frac{1}{\left(\frac{\pi r^2}{4}\right)} \sum \frac{y}{2} . y\delta x$$

$$= \frac{1}{\left(\frac{\pi r^2}{4}\right)} \int_0^r \frac{y}{2} . ydx$$

$$= \frac{2}{\pi r^2} \int_0^r r^2 - x^2 dx, \quad \text{since } x^2 + y^2 = r^2$$

$$= \frac{2}{\pi r^2} \left[r^2 x - \frac{1}{3} x^3 \right]_0^r = \frac{2}{\pi r^2} \left(\frac{2r^3}{3} \right) = \frac{4r}{3\pi}$$

(b)



Considering horizontal strips as shown in the diagram,

each strip has area $x\delta y$, and $\overline{y}$ for each strip is y.

Then $\overline{y} = \frac{1}{\left(\frac{\pi r^2}{4}\right)} \int_0^r y . xdx$

$$= \frac{4}{\pi r^2} \int_0^r \sqrt{r^2 - x^2} . xdx$$

As $\frac{d}{dx}(r^2 - x^2) = -2x$ and we have an x in the integrand,

we can make the substitution $u = r^2 - x^2$, so that $du = -2xdx$

and $\overline{y} = \frac{4}{\pi r^2}\int_{r^2}^{0} \sqrt{u}.\left(-\frac{1}{2}\right) du = \frac{2}{\pi r^2}\int_{0}^{r^2} \sqrt{u}.\, du$

$= \frac{2}{\pi r^2}\left[\frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_{0}^{r^2} = \frac{4r}{3\pi}$, as before

(2) The region between the curve $y = x^3 - x^2$ and the x-axis is rotated by $360°$ about the x-axis. Find the centre of mass of the solid of revolution obtained.

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Solution

By symmetry, $\bar{y} = 0$

$y = x^3 - x^2 = x^2(x-1)$ meets the x-axis at $x = 0$ & $x = 1$

$V\bar{x} = \int_0^1 x(\pi y^2 dx)$, where $V = \int_0^1 \pi y^2 dx$

[$\pi y^2 dx$ is the volume of the disc of width dx at position x]

Thus $V = \pi \int_0^1 x^4(x-1)^2 dx = \pi \int_0^1 x^6 - 2x^5 + x^4\, dx$

$= \pi[\frac{1}{7}x^7 - \frac{2}{6}x^6 + \frac{1}{5}x^5]_0^1$

$= \pi\left(\frac{1}{7} - \frac{2}{6} + \frac{1}{5}\right) = \frac{(30-70+42)\pi}{210} = \frac{\pi}{105}$ $units^3$

And $\bar{x} = 105 \int_0^1 x^7 - 2x^6 + x^5\, dx$

$= 105[\frac{1}{8}x^8 - \frac{2}{7}x^7 + \frac{1}{6}x^6]_0^1$

$= 105\left(\frac{1}{8} - \frac{2}{7} + \frac{1}{6}\right) = \frac{105(21-48+28)}{168}$

$= \frac{105}{168} = \frac{35}{56} = \frac{5}{8} = 0.625$

Thus the centre of mass is $(0.625, 0)$

(3) Find the centre of mass of the region between the curve $y = x^3 - x^2$ and the x-axis.

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$y = x^3 - x^2$ and the x-axis.

Solution

$y = x^3 - x^2 = x^2(x-1)$ meets the x-axis at $x = 0$ & $x = 1$

Note that the curve lies beneath the x-axis.

Total weight (signed area)

$= \int_0^1 x^3 - x^2\,dx = \left[\frac{1}{4}x^4 - \frac{1}{3}x^3\right]_0^1 = \left(\frac{1}{4} - \frac{1}{3}\right) = -\frac{1}{12}$

$-\frac{1}{12}\bar{x} = \int_0^1 x(x^3 - x^2\,dx) = \left[\frac{1}{5}x^5 - \frac{1}{4}x^4\right]_0^1 = \left(\frac{1}{5} - \frac{1}{4}\right) = -\frac{1}{20}$

so that $\bar{x} = 0.6$

And $-\frac{1}{12}\bar{y} = \int_0^1 \frac{y}{2}(x^3 - x^2\,dx) = \frac{1}{2}\int_0^1 (x^3 - x^2)^2 dx$

$= \frac{1}{2}\int_0^1 x^6 + x^4 - 2x^5\,dx$

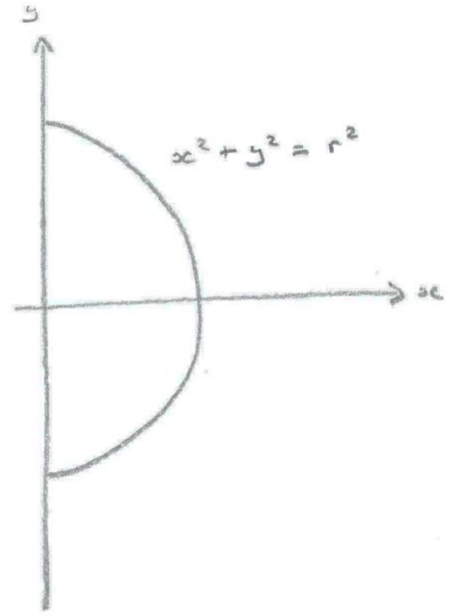
$= \frac{1}{2}\left[\frac{1}{7}x^7 + \frac{1}{5}x^5 - \frac{2}{6}x^6\right]_0^1$

$= \frac{1}{2}\left(\frac{1}{7} + \frac{1}{5} - \frac{2}{6}\right)$

so that $\bar{y} = \frac{-6(30+42-70)}{210} = -\frac{2}{35} = -0.0571$ (3sf)

Thus the centre of mass is $(0.6, -0.0571)$

(4) Find the centre of mass of the semi-circular lamina shown in the diagram.



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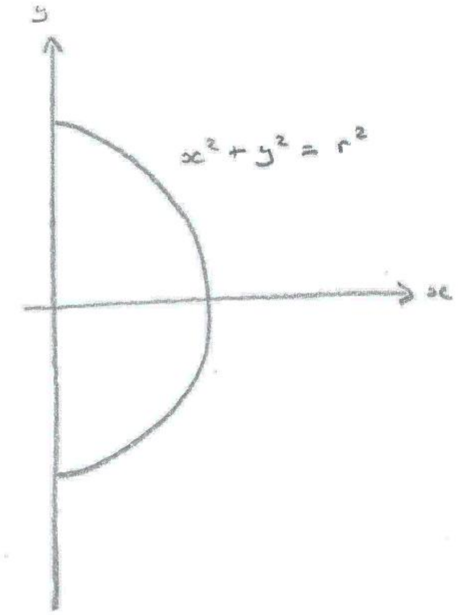

Solution

By symmetry, we need only consider the top half

$$\bar{x} = \frac{1}{\frac{1}{4}\pi r^2}\int_0^r xy\,dx = \frac{4}{\pi r^2}\int_0^r x\sqrt{r^2 - x^2}\,dx$$

$$= \frac{-2}{\pi r^2}\int_0^r (-2x)\sqrt{r^2 - x^2}\;dx$$

$$= \frac{-2}{\pi r^2}\left[\frac{(r^2 - x^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^r = -\frac{4}{3\pi r^2}(0 - r^3) = \frac{4r}{3\pi}$$

(5) Show that the centre of mass of a parallelogram is at the intersection of the diagonals, by finding the centre of mass of two triangles, given the result that the diagonals bisect each other.

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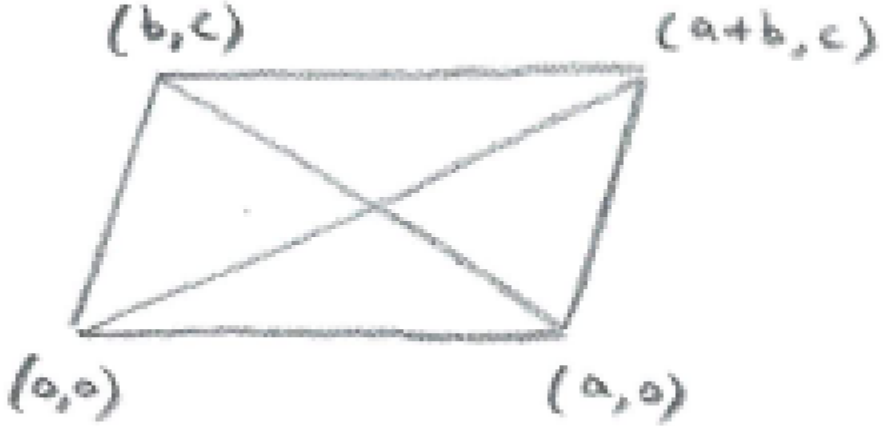


Let triangle 1 have corners

(0,0) , (a,0) & (b,c)

(and triangle 2 be the other half of the parallelogram).

$$CoM_1 = \begin{pmatrix} \frac{1}{3}(0+a+b) \\ \frac{1}{3}(0+0+c) \end{pmatrix} \ \& \ CoM_2 = \begin{pmatrix} \frac{1}{3}(a+b+[a+b]) \\ \frac{1}{3}(0+c+c) \end{pmatrix}$$

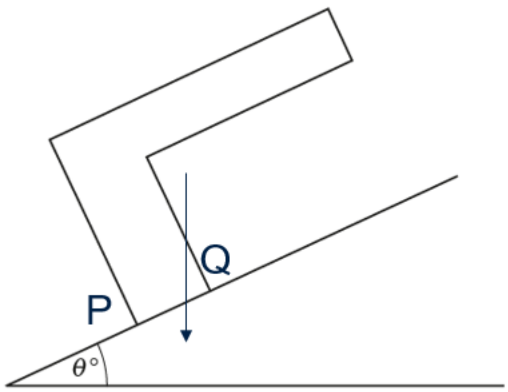
Then centre of mass of parallelogram $= \frac{1}{2}\ (COM_1 + COM_2)$

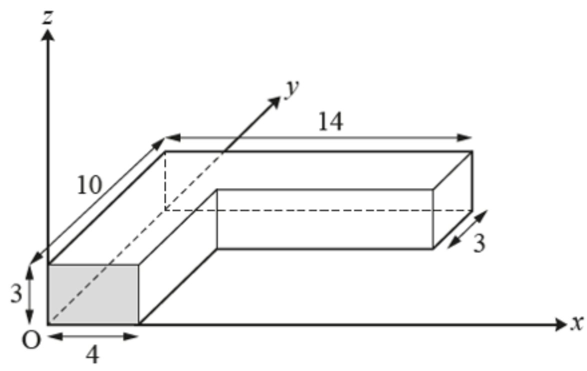
$$= \frac{1}{6}\begin{pmatrix} 3a+3b \\ 3c \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a+b \\ c \end{pmatrix}$$

ie the mid-point of the diagonal from (0,0) to (a+b,c)

(6) An object lies on a slope as shown in the left-hand diagram below. The dimensions of the object are shown in the right-hand diagram. The shaded end is in contact with the slope and the 4cm edge is along the line of greatest slope.

Given that the surface of the slope is sufficiently rough that the object will not slip, determine the minimum and maximum possible values of θ for which the object does not topple.





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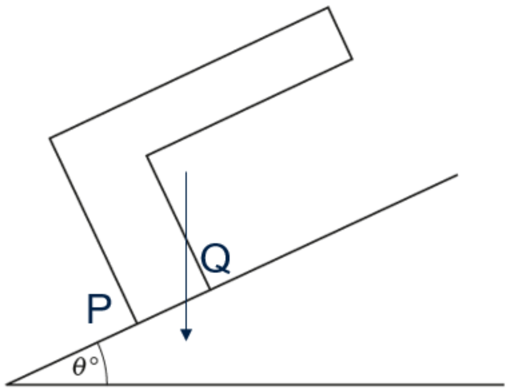

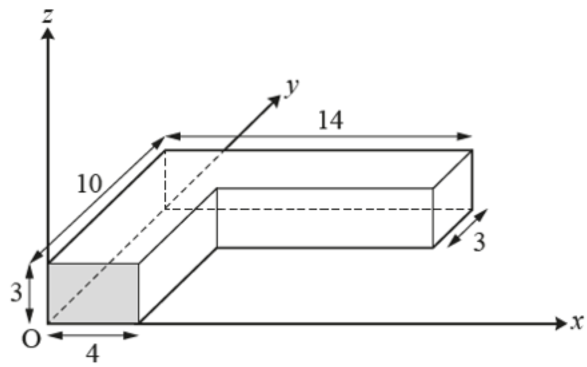

Solution

To find the centre of mass of the object:

Split the ornament into 2 parts: A is the lefthand cuboid, with dimensions $4 \times 10 \times 3$ (and volume 120); B is the righthand cuboid with dimensions $10 \times 3 \times 3$ (and volume 90)

The CoM of the object is:

$$\frac{1}{(120+90)}\left[120\begin{pmatrix}2\\5\\1.5\end{pmatrix}+90\begin{pmatrix}9\\8.5\\1.5\end{pmatrix}\right]=\frac{10}{210}\begin{pmatrix}105\\136.5\\31.5\end{pmatrix}=\begin{pmatrix}5\\6.5\\1.5\end{pmatrix}$$

For sufficiently large values of θ, the object may topple about P, whilst for sufficiently small values it may topple about Q.

For P, the critical point occurs when the CoM lies vertically above P; ie when $\tan\theta = \frac{5}{6.5}$, and $\theta \approx 37.569 = 37.6°$ (3sf)

For Q, the critical point occurs when the CoM lies vertically above Q; ie when $\tan\theta = \frac{(5-4)}{6.5}$, and $\theta \approx 8.7462 = 8.75°$ (3sf)

Thus the minimum and maximum possible values of θ are 8.75° and 37.6°.