Centre of Mass - Part 2 (12 pages; 7/8/17)

(13) Centre of mass of triangular lamina (proof) – Part 1

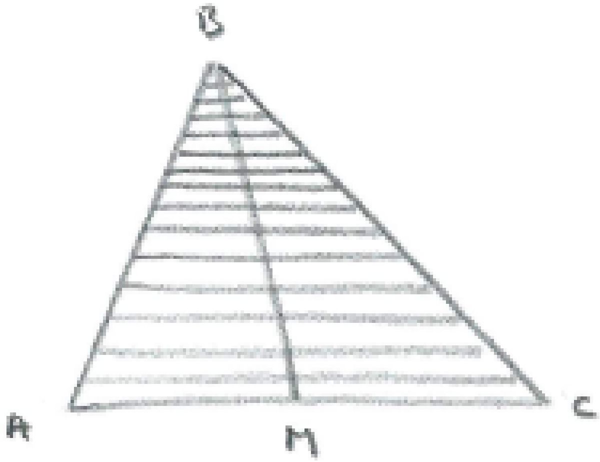

Figure 16

Referring to Figure 16, the centre of mass of each strip lies halfway along it; ie on the median BM. (If we consider, for example, the strip $A'C'$ starting halfway along AB, and intersecting AB, MB & CB at A', M'& C', then ABM & $A'BM'$ are similar triangles, so that $A'M'$ is a half of AM. In the same way, $M'C'$ is a half of MC. Hence $A'M' = M'C'$.)

Treating the lamina as a composite of an infinite number of these strips, its centre of mass will be that of an infinite number of point masses lying on BM. Thus the centre of mass of the lamina must lie on BM, and similarly it must lie on the other two medians. This shows that the 3 medians must meet at a point, which is the centre of mass of the lamina.

It is shown below that the centre of mass lies two-thirds of the way along any of the medians, from the vertex.

(14) Centre of mass of triangular lamina (proof) – Part 2 (Distance along the medians)

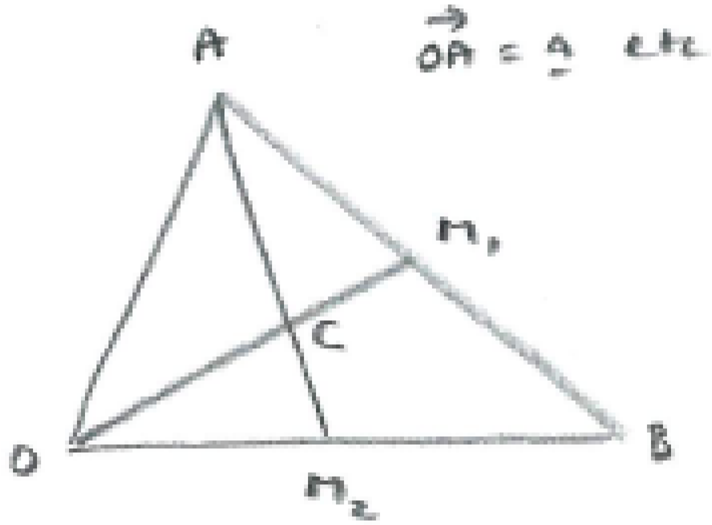

Figure 17

Let $\overrightarrow{OC} = \lambda\overrightarrow{OM_1}$ & $\overrightarrow{AC} = \mu\overrightarrow{AM_2}$ (1)

[standard technique: represents the fact that C lies on the line OM_1]

Also, $\overrightarrow{OC} = \underline{a} + \overrightarrow{AC}$ (2) [standard technique: 2 ways of getting to the same place]

Substitute (1) into (2) ⇒ $\lambda\overrightarrow{OM_1} = \underline{a} + \mu\overrightarrow{AM_2}$ (3)

Now, $\overrightarrow{OM_1} = ½\,(\underline{a} + \underline{b})$ & $\overrightarrow{AM_2} = ½\,\underline{b} - \underline{a}$ (4)

Substitute (4) into (3) ⇒ $½\,\lambda\,(\underline{a} + \underline{b}) = \underline{a} + \mu\,(½\,\underline{b} - \underline{a})$ (5)

$\Rightarrow (\frac{\lambda}{2} + \mu - 1)\,\underline{a} + (\frac{\lambda}{2} - \frac{\mu}{2})\,\underline{b} = 0$

Provided $\underline{a}$ & $\underline{b}$ are not parallel, there is only one way of expressing a vector as a combination of $\underline{a}$ & $\underline{b}$

In this case, $\frac{\lambda}{2} + \mu - 1 = 0$ & $\frac{\lambda}{2} - \frac{\mu}{2} = 0$ (6)

[standard technique: equivalent to equating coefficients of $\underline{a}$ & of $\underline{b}$ in (5)]

Then (6) $\Rightarrow \lambda = \mu = \frac{2}{3}$

ie the centre of mass lies two-thirds of the way along any of the medians, from the relevant vertex.

(15) Centre of mass of triangular lamina (prrof) – Part 3: Location of the centre of mass in terms of the coordinates of the vertices

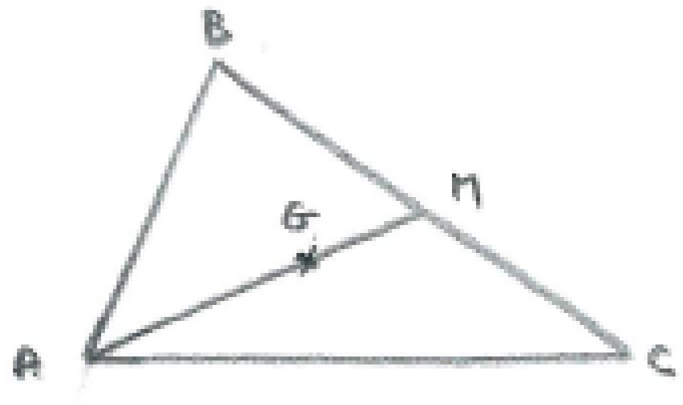

Figure 18

$$\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AG}$$

$$= \underline{a} + \frac{2}{3}\overrightarrow{AM}$$

$$= \underline{a} + \frac{2}{3}.\tfrac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$$

$$= \underline{a} + \frac{1}{3}[(\underline{b} - \underline{a}) + (\underline{c} - \underline{a})]$$

$$= \frac{1}{3}(\underline{a} + \underline{b} + \underline{c})$$

So if $\underline{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ etc, $\overrightarrow{OG} = \begin{pmatrix} \frac{1}{3}(a_x + b_x + c_x) \\ \frac{1}{3}(a_y + b_y + c_y) \end{pmatrix}$

(16) Centre of mass of triangular lamina (proof) – Part 4 : Right-angled Triangle

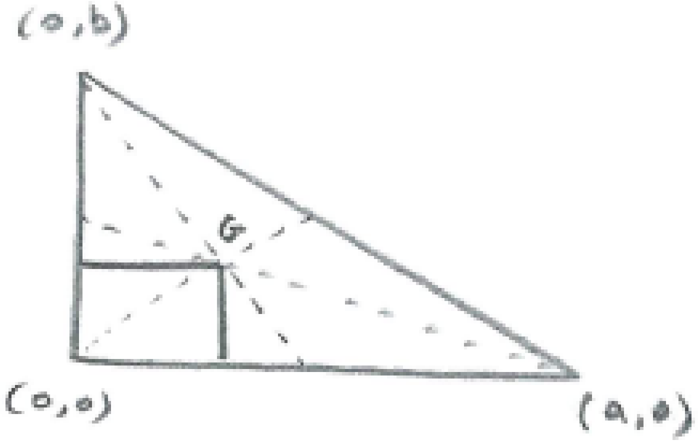

Figure 19

G is at $(\frac{1}{3}(0 + a + 0),\ \frac{1}{3}(0 + 0 + b)) = (\frac{a}{3}, \frac{b}{3})$

Note: The $\frac{1}{3}$ in this formula is nothing to do with G being $\frac{1}{3}$ of the way along the median (from the side towards the vertex).

This result can also be obtained by integration.

(17) Centre of mass of a uniform sector (proof by integration)

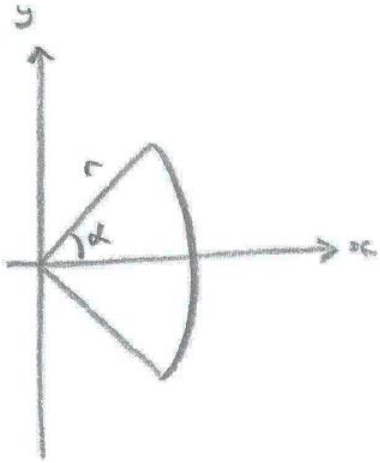

Figure 20

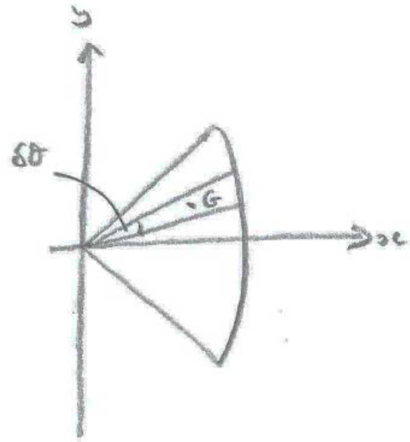

Figure 21

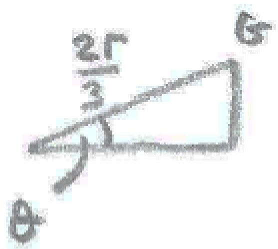

Figure 22

By symmetry, we need only consider the top half of Figure 20.

Centre of mass of thin wedge (approx. triangle) is at G: $\frac{2}{3}r$ from O:

$\overline{x}$ (wedge)$=\frac{2}{3}r\cos\theta$

Area of wedge $=\frac{1}{2}r^2\delta\theta$

Area of (top half) of sector $= \frac{1}{2}r^2\alpha$

$\bar{x}$ (half sector) $= \frac{1}{\frac{1}{2}r^2\alpha}\int_0^\alpha (\frac{2}{3}r\cos\theta)(\frac{1}{2}r^2 d\theta)$

$= \frac{2r}{3\alpha}\int_0^\alpha cos\theta\ d\theta = \frac{2r}{3\alpha}[sin\theta]_0^\alpha = \frac{2rsin\alpha}{3\alpha}$

(18) Centre of mass of a uniform circular arc (proof by integration)

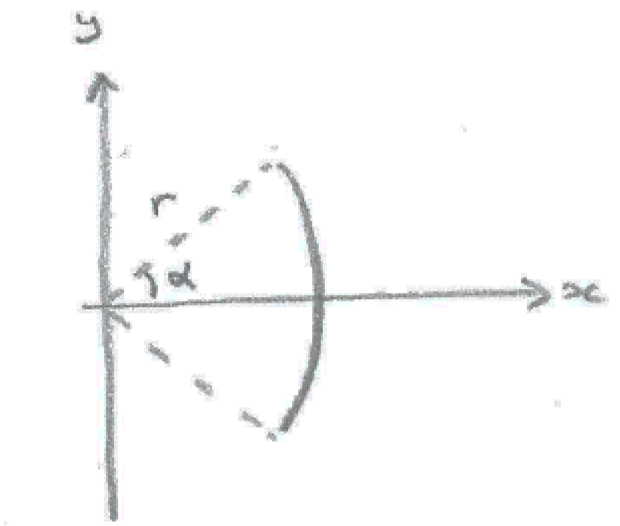

Figure 23

By symmetry we need only consider the top half of Figure 23.

$\bar{x}$ (element of arc)$= r\cos\theta$ (where θ is as in (17))

length of element $= r\delta\theta$

Length of (top half) of arc $= r\alpha$

$\bar{x}$ (top half of wire) $= \frac{1}{r\alpha}\int_0^\alpha (r\cos\theta)(r d\theta)$

$= \frac{r}{\alpha}[sin\theta]_0^{\alpha} = \frac{rsin\alpha}{\alpha}$

(19) Centre of mass of a general lamina (proof by integration)

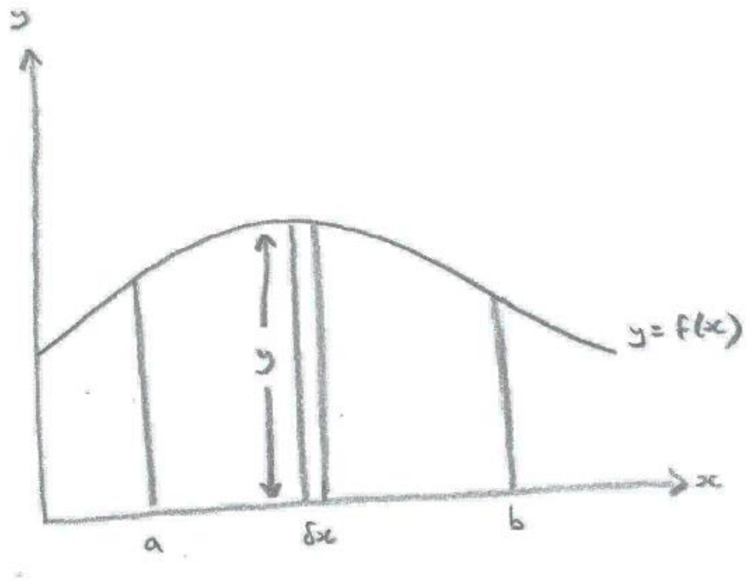

Figure 24

$\overline{x} = \frac{1}{A}\int_a^b x(ydx)$

where $A = \int_a^b y\, dx$

(area is proxy for mass)

$\overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}y(ydx)$

Note: In some cases it may be easier to integrate wrt y;

ie $\overline{y} = \frac{1}{A}\int_c^d y(xdy)$ etc

(20) Centre of mass of a right-angled triangular lamina (proof by integration)

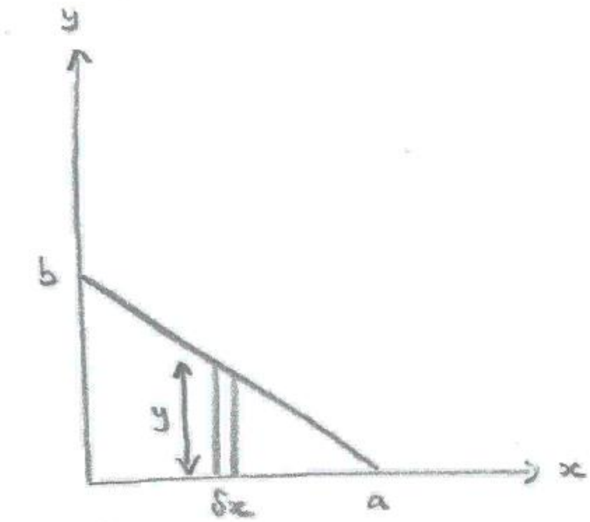

Figure 25

$\overline{x} = \frac{1}{A}\int_0^a x(ydx)$; where $y = -\frac{b}{a}x + b$ (or $\frac{x}{a} + \frac{y}{b} = 1\ etc$)

and $A = \frac{1}{2}ab$

So $\overline{x} = \frac{2}{ab}\int_0^a -\frac{b}{a}x^2 + bx\ \ dx$

$= \frac{2}{ab}\left[-\frac{b}{a}(\frac{x^3}{3}) + b(\frac{x^2}{2})\right]\begin{matrix} a \\ 0 \end{matrix}$

$= \frac{2}{ab}\left(-\frac{ba^3}{3a} + \frac{ba^2}{2}\right) = \frac{2}{ab}\left(\frac{ba^2}{6}\right) = \frac{a}{3}$

$\overline{y} = \frac{1}{A}\int_0^a \frac{1}{2}y(ydx)$; where $y = -\frac{b}{a}x + b$ & $A = \frac{1}{2}ab$

So $\overline{y} = \frac{2}{ab}\int_0^a \frac{1}{2}\left(-\frac{b}{a}x + b\right)^2\ \ dx$

$= \frac{b}{a}\int_0^a \left(-\frac{x}{a} + 1\right)^2 dx = \frac{b}{a}\int_0^a \frac{x^2}{a^2} - \frac{2x}{a} + 1\, dx$

$= \frac{b}{a}\left[\frac{x^3}{3a^2} - \frac{2x^2}{2a} + x\right]_0^a$

$= \frac{b}{a}\left(\frac{a}{3} - a + a\right) = \frac{b}{3}$

$\overline{x} = \frac{a}{3}$; $\overline{y} = \frac{b}{3}$

(21) Centre of mass of cone (proof by integration)

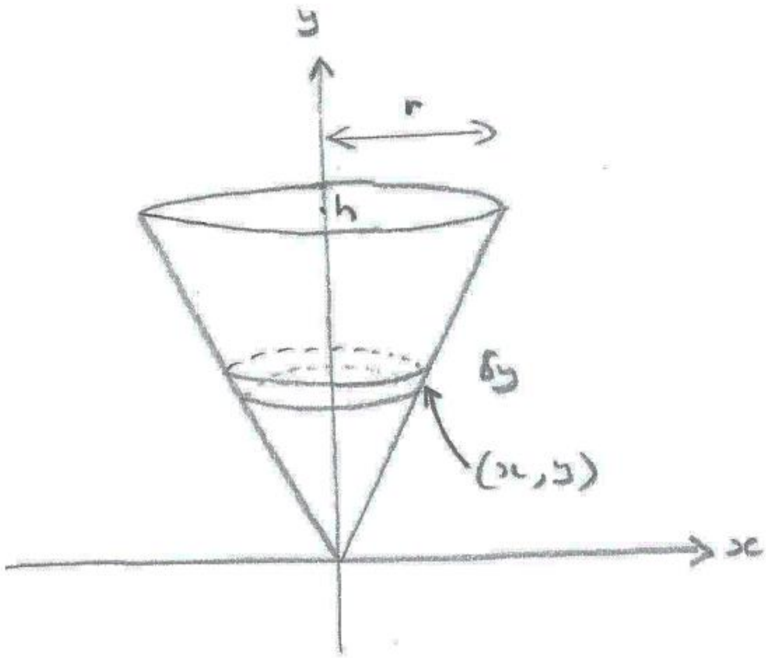

Figure 26

$y = \frac{h}{r}x \Rightarrow x^2 = \left(\frac{ry}{h}\right)^2$

Volume $= \int_0^h \pi x^2\, dy = \int_0^h \pi \left(\frac{ry}{h}\right)^2 dy$

$= \frac{\pi r^2}{h^2}\int_0^h y^2 dy = \frac{\pi r^2}{h^2}\left[\frac{1}{3}y^3\right]_0^h$

$= \frac{\pi r^2}{h^2}\left(\frac{1}{3}h^3\right) = \frac{\pi r^2 h}{3} \quad (\frac{1}{3} \times base\ area \times height)$

$\bar{y} = \frac{1}{V}\int_0^h y(\pi x^2\, dy) = \frac{1}{V}\int_0^h y\,\pi\left(\frac{ry}{h}\right)^2 dy$

$= \frac{\pi r^2}{Vh^2}\int_0^h y^3 dy = \frac{\pi r^2}{Vh^2}\left[\frac{1}{4}y^4\right]_0^h$

$= \frac{\pi r^2}{Vh^2}\left(\frac{1}{4}h^4\right) = \frac{\pi r^2 h^2}{4V} = \frac{\pi r^2 h^2}{4\left(\frac{\pi r^2 h}{3}\right)} = \frac{3h}{4}$

(22) Centre of mass of conical shell

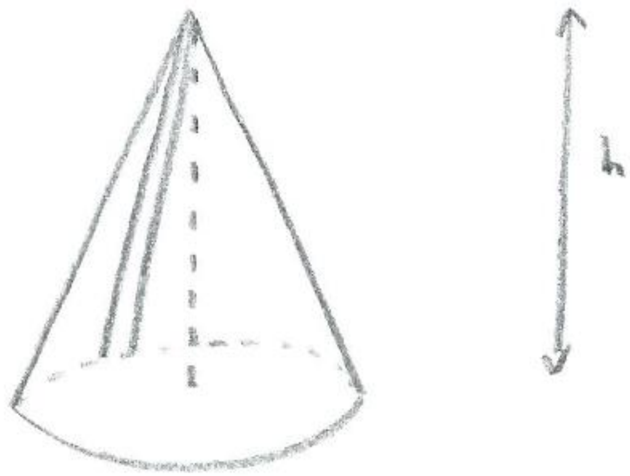

Figure 27

Centre of mass of strip is $\frac{2}{3}$ along from the vertex.

So centre of mass is $\frac{2}{3}h$ below the vertex (by similar triangles);

or $\frac{h}{3}$ above the base

(23) Centre of mass of hemispherical shell (proof by integration)

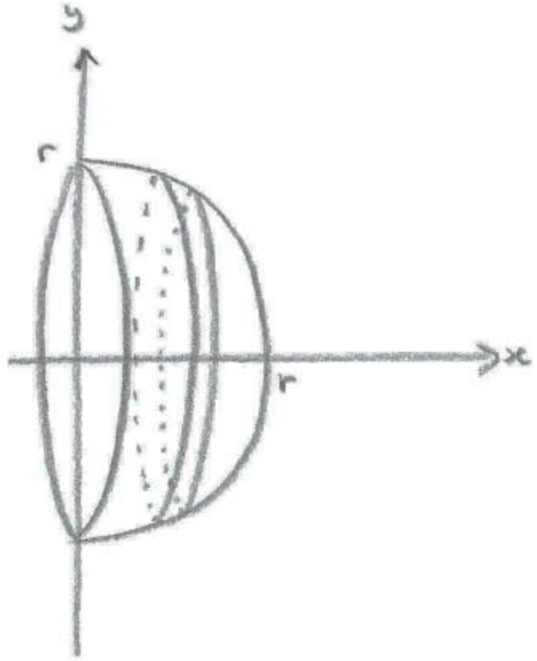

Figure 28

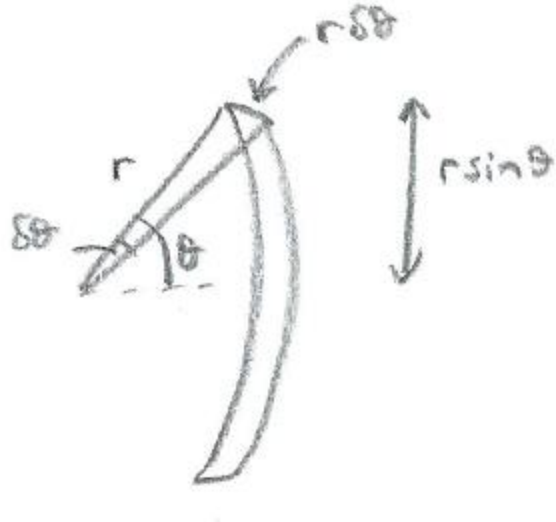

Figure 29

Surface area of sphere $= 4\pi r^2$

$\overline{x} = \frac{1}{2\pi r^2}\int_0^{\frac{\pi}{2}}(rcos\theta)(2\pi[rsin\theta][rd\theta])$

$= r\int_0^{\frac{\pi}{2}} cos\theta sin\theta \, d\theta$

$= \frac{r}{2}\int_0^{\frac{\pi}{2}} sin2\theta \, d\theta$

$$= \frac{r}{2}\left[-\frac{1}{2}cos2\theta\right]_{0}^{\frac{\pi}{2}}$$

$$= \frac{-r}{4}(cos\pi - cos0) = \frac{-r}{4}(-1-1) = \frac{r}{2}$$