Centre of Mass - Part 1 (13 pages; 7/8/17)

[See separate Contents document; Part 2 contains proofs]

(1) Objects for which the centre of mass may be determined

(a) Collection of point masses (1D, 2D or 3D)

(b) Solid (or hollow) object

(c) Lamina (a thin plane object – eg a piece of paper)

(d) A rod (of relatively small radius)

(e) A composite body (a combination of (a)-(d))

(2) Centre of mass of a collection of point masses

Suppose that there are 3 point masses with coordinates and weights as shown in Figure 1 (the line has no weight).

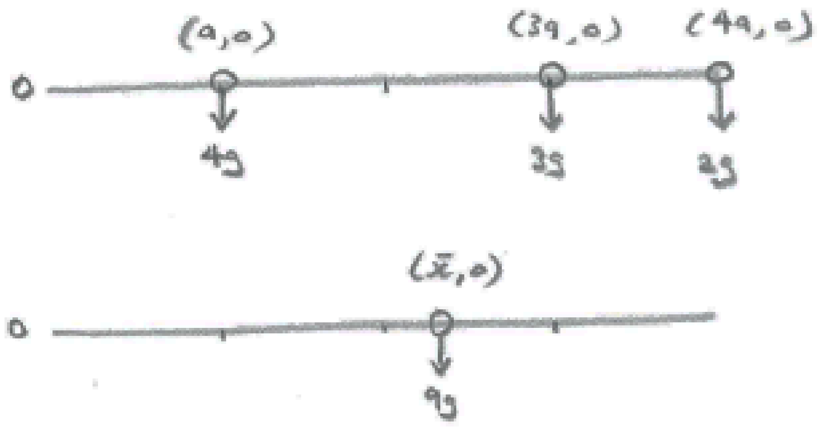

Figure 1

Consider a point mass having a weight equal to the total of the weights of the point masses (ie $9g$ in this case).

The centre of mass of the original point masses can be considered to be the location of the $9g$ point mass, such that it has the same moment about O as that of the original point masses.

Total (clockwise) moment of original point masses

$= 4g(a) + 3g(3a) + 2g(4a) = 21ga$

Let centre of mass be at $(\overline{x}\ , 0)$

Then moment of $9g$ point mass $= (9\text{g})\overline{x}$

Hence $21ga = (9\text{g})\overline{x}$ and so $\overline{x} = \frac{7}{3}a$

In general, if the masses $m_1,\ m_2,\ \ldots, m_n$ are positioned at the points $(x_1, 0)$, $(x_2, 0)$, $\ldots, (x_n, 0)$, then

$\text{M}\overline{x} = \sum_{i=1}^{n} m_i x_i$, where $\text{M} = \sum_{i=1}^{n} m_i$

Alternative approach

The centre of mass can also be thought of as the weighted average of the positions of the individual point masses, where the weights are just the relative masses.

Thus $\overline{x} = \sum_{i=1}^{n} (\frac{m_i}{M}) x_i$

(3) Collection of point masses in 2D

$\overline{x}$ & $\overline{y}$ can be found separately:

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} \sum_{i=1}^{n} m_i x_i \\ \sum_{i=1}^{n} m_i y_i \end{pmatrix}$$

This can be extended simply to 3D.

Example: Find the centre of mass of the following set of point masses

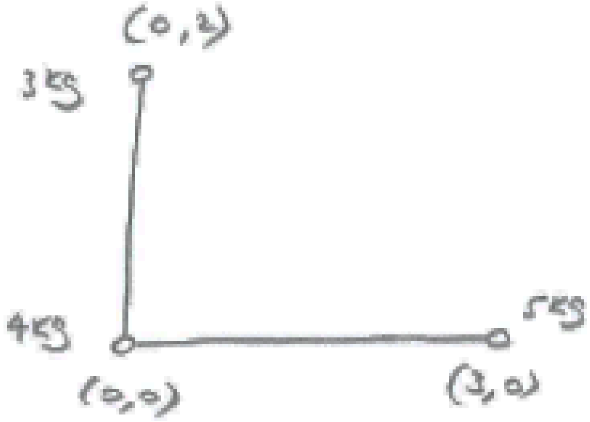

Figure 2

Solution

$(3+4+5)\binom{\bar{x}}{\bar{y}} = 3\binom{0}{2} + 4\binom{0}{0} + 5\binom{3}{0}$

$\Rightarrow 12\binom{\bar{x}}{\bar{y}} = \binom{15}{6}$

$\Rightarrow \binom{\bar{x}}{\bar{y}} = \binom{1.25}{0.5}$

Note the following advantages of using column vectors, rather than determining $\bar{x}$ and $\bar{y}$ separately:

(i) There is less chance of mixing up the weights and the distances.

(ii) There is less chance of missing a component for $x\ or\ y$.

(iii) It provides a check that no component has been missed, if the centre of mass for either $x\ or\ y$ can be deduced from symmetry.

(4) Centre of mass of a solid (or hollow) object

The first consideration will be whether there is any symmetry (when the centre of mass will lie on the axis of symmetry).

Otherwise, the formulae booklet may need to be consulted. The main ones are listed here. These formulae are generally derived by integration (see Part 2).

Solid hemisphere of radius r : $\frac{3}{8}r$ from the centre

Hollow hemisphere of radius r : $\frac{1}{2}r$ from the centre

Solid cone or pyramid of height h: $\frac{1}{4}h$ above the base

Hollow cone or pyramid of height h: $\frac{1}{3}h$ above the base

(5) Centre of mass of a lamina

Once again, symmetry needs to be considered. The standard formulae are given below. These can be derived by integration, or sometimes using vectors (see Part 2).

Triangular lamina: $\frac{2}{3}$ along the median from any vertex (the median is the line from the vertex to the mid-point of the opposite side).

Sector of circle of radius $\boldsymbol{r}$, with angle at the centre of 2α (in radians): $\frac{2rsin\alpha}{3\alpha}$ from the centre

[Special case: a **semi-circular lamina of radius $\boldsymbol{r}$**: $\frac{4r}{3\pi}$ from the centre.]

(6) Centre of mass of a uniform triangular lamina

There are several ways of locating this (see Part 2 for proofs).

(a) The average of the coordinates of the vertices.

(b) Special case of a right-angled triangular lamina: $\frac{1}{3}$ of the way along the sides creating the right angle, starting at the right angle (see Figure 3) [this can be demonstrated by applying (a)].

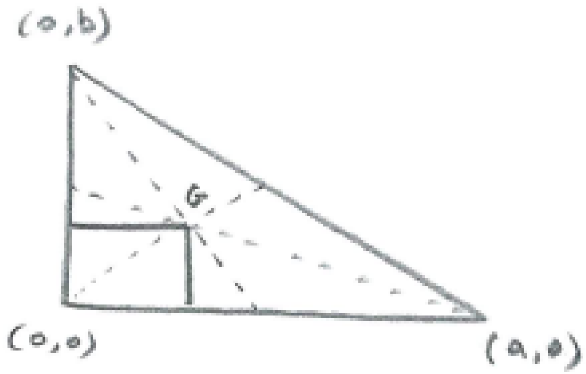

Figure 3

(c) $\frac{2}{3}$ of the way along the median from any one of the vertices (a median connects a vertex with the midpoint of the opposite side, and the 3 medians meet at the centre of mass; this is true for all triangles, but is illustrated for a right-angled triangle in Figure 3.

Notes

(i) The $\frac{1}{3}$ in (b) is nothing to do with G (the centre of mass) being $\frac{1}{3}$ of the way along the median from the side to the vertex).

(ii) Be careful to distinguish between:

(a) a triangular lamina

(b) a triangular framework or rods (see below)

(c) point masses at the vertices of a triangle

Example: Find the position of the centre of mass of the uniform lamina shown

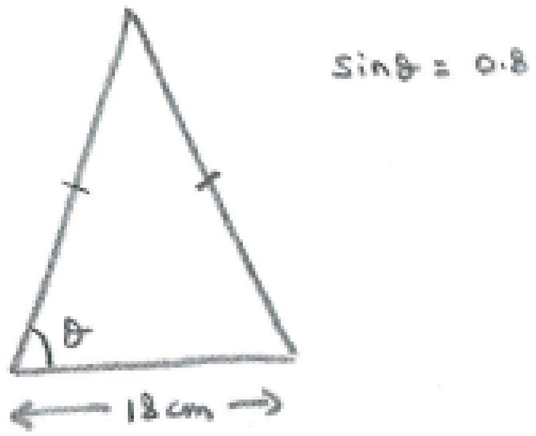

Figure 4

Method 1

Taking the origin to be the bottom left-hand corner.

By symmetry, $\overline{x} = 9$

$\overline{y}$ will be $\frac{2}{3}$ of the way down the vertical median

$\sin\theta = 0.8 = \frac{4}{5} \Rightarrow \tan\theta = \frac{4}{3}$

$\Rightarrow$ height of triangle $= \frac{4}{3}$ x $9 = 12$

Hence $\overline{y} = 4$

Method 2

As the height is 12, the coordinates of the vertices are (0,0) , (18,0) & (9,12)

Hence $\overline{x} = \frac{1}{3}(0 + 18 + 9) = 9$

And $\overline{y} = \frac{1}{3}(0 + 0 + 12) = 4$

Exercise: Where would the centre of mass be if all the weight of the lamina was concentrated equally at the 3 corners?

$\overline{x} = \frac{1}{3}(0) + \frac{1}{3}(9) + \frac{1}{3}(18)$

$= 9$ (as expected, by symmetry)

$\overline{y} = \frac{1}{3}(0) + \frac{1}{3}(12) + \frac{1}{3}(0) = 4$ (as before)

(7) Centre of mass of a uniform sector

The centre of mass of a uniform sector of a circle of radius r and angle 2α is $\frac{2r sin\alpha}{3\alpha}$ from the centre (see Part 2 for proof).

As $\alpha \to 0$, $\frac{2r sin\alpha}{3\alpha} \to \frac{2r}{3}$; the sector tends to a triangle (see Figure 7) and the position of the centre of mass tends to $\frac{2}{3}r$ from the centre (ie $\frac{2}{3}$ of the way along the median from the vertex).

Special case: semi-circle ($\alpha = \frac{\pi}{2}$): $\overline{x} = \frac{4r}{3\pi}$

Figures 5 & 6

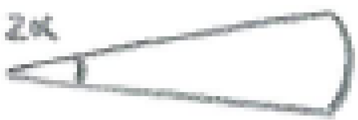

Figure 7

(8) Centre of mass of a rod

If the rod can be assumed to be uniform (ie its mass is distributed uniformly), then the centre of mass will simply be at the mid-point.

If this isn't the case, then integration would be needed.

(9) Centre of mass of a uniform circular arc

The centre of mass of a uniform circular arc, where the radius is r and the angle is 2α is $\frac{rsin\alpha}{\alpha}$ from the centre (see Part 2 for proof).

As $\alpha \to 0$, $\frac{rsin\alpha}{\alpha} \to r$; the arc tends to a point, and the position of the centre of mass tends to r from the centre.

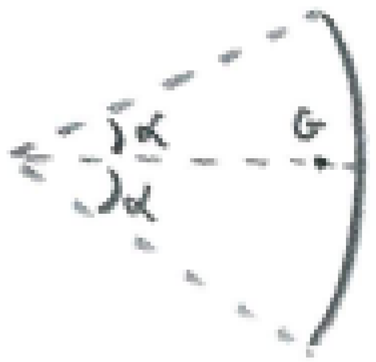

Figure 8

(10) Centre of mass of a composite body

The composite body could in theory be made up of any combination of solids, laminas or rods (though in practice it tends to be either a combination of solids, or a combination of laminas, or a combination of rods (a 'framework').

The procedure is to treat the composite body as a collection of point masses, located at the centre of mass of each component.

Volume, area or length may be used as a proxy for mass (in the case of solids, laminas & rods, respectively).

Example (Lamina)

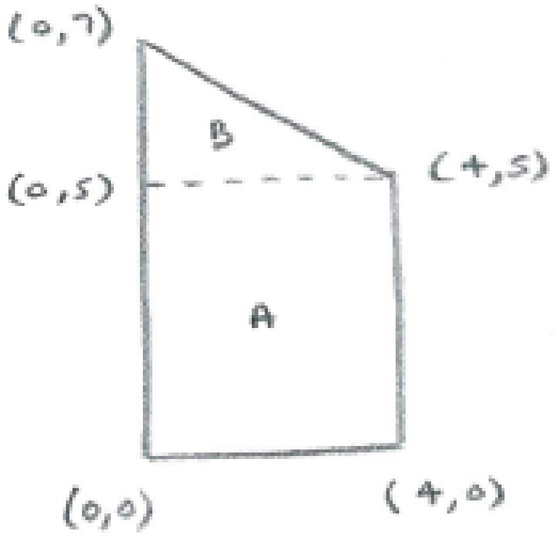

Figure 9

CoM of A: $\begin{pmatrix} 2 \\ \frac{5}{2} \end{pmatrix}$

CoM of B: $\begin{pmatrix} \frac{1}{3}(0+4+0) \\ \frac{1}{3}\ (5+5+7) \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{17}{3} \end{pmatrix}$

Area of A $= 20$; Area of B $= 1/2\ (4)(2) = 4$

CoM of trapezium $= \frac{1}{24}\begin{pmatrix} 20(2) + 4(\frac{4}{3}) \\ 20\left(\frac{5}{2}\right) + 4(\frac{17}{3}) \end{pmatrix} = \begin{pmatrix} \frac{17}{9} \\ \frac{109}{36} \end{pmatrix}$

Example (Framework)

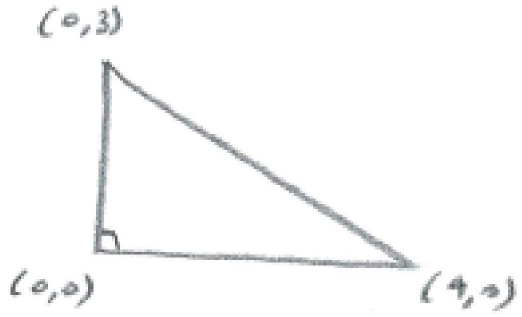

Figure 10

The centres of mass of the 3 sides are:

$\begin{pmatrix}0\\1.5\end{pmatrix}$, $\begin{pmatrix}2\\0\end{pmatrix}$ & $\begin{pmatrix}2\\1.5\end{pmatrix}$

The centre of mass of the framework

$$= \frac{1}{(3+4+5)}\begin{pmatrix}3(0)+4(2)+5(2)\\3(1.5)+4(0)+5(1.5)\end{pmatrix} = \begin{pmatrix}1.5\\1\end{pmatrix}$$

Example: Centre of mass of a rectangular lamina with a square cut out of it

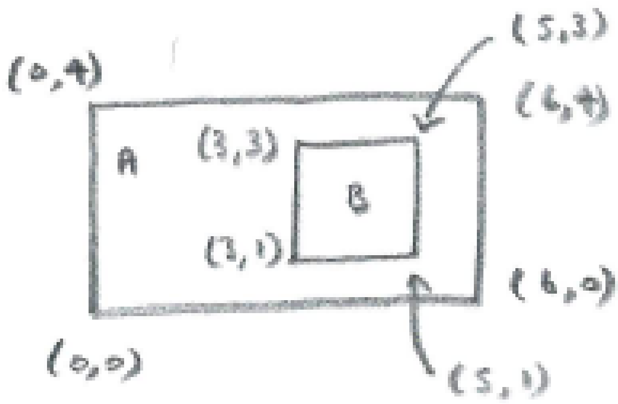

Figure 11

Let the centre of mass of the required shape be $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$

The centre of mass of A+B $= \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

The centre of mass of B $= \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Area of A+B $= 24$

Area of B $= 4$

Then $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \frac{1}{24} \begin{pmatrix} 4(4) + 20\bar{x} \\ 4(2) + 20\bar{y} \end{pmatrix}$

Hence $\begin{pmatrix} 72 \\ 48 \end{pmatrix} = \begin{pmatrix} 16 + 20\bar{x} \\ 8 + 20\bar{y} \end{pmatrix}$

So that $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} 2.8 \\ 2 \end{pmatrix}$

Alternative approach

The missing square can be treated as having a negative weight, so that $20\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = 24\begin{pmatrix} 3 \\ 2 \end{pmatrix} - 4\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ etc

(11) Hanging a lamina from one of its corners

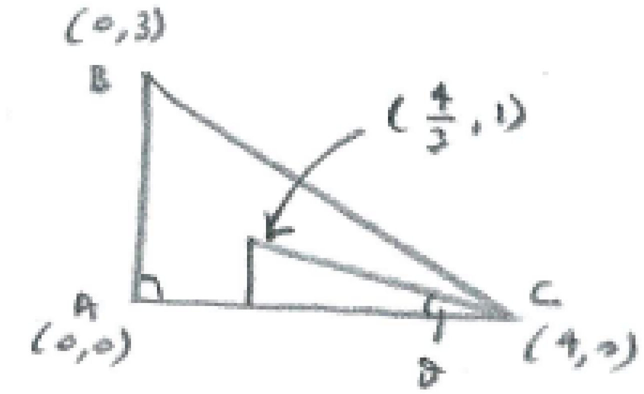

Figure 12

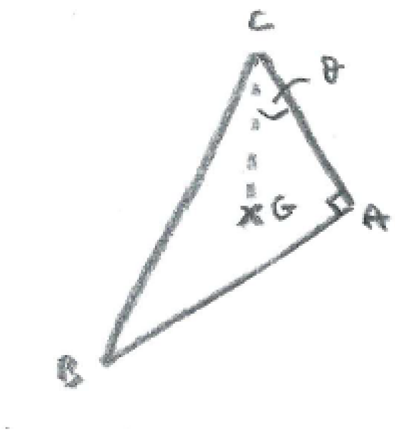

Figure 13

The centre of mass of the above lamina can be found to be at $(\frac{4}{3}, 1)$. When the lamina is hung from C, the centre of mass G will be directly below C: were it not to be, then taking moments about C would produce a non-zero moment of the weight at G.

To find the angle θ that the side AC makes with the vertical, it can be easier to draw in the line from G to C, without actually showing the triangle in its hanging position (this has been done in an exam mark scheme, for example).

$tan\theta = \frac{1}{\left(\frac{4}{3}\right)} = \frac{3}{4} \Rightarrow \theta = 36.9°(3sf)$

(12) Toppling: Lamina on a rough slope

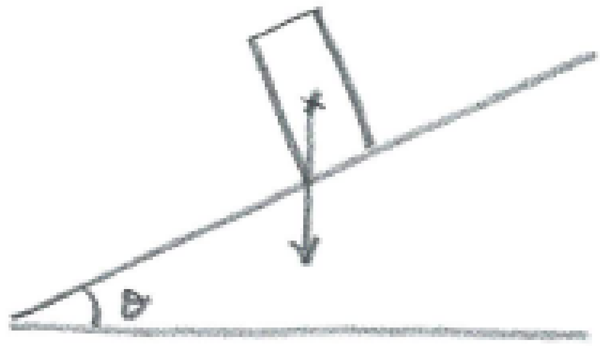

Figure 14

Block is on the point of toppling. If block is 4cm x 6cm, find $tan\theta$.

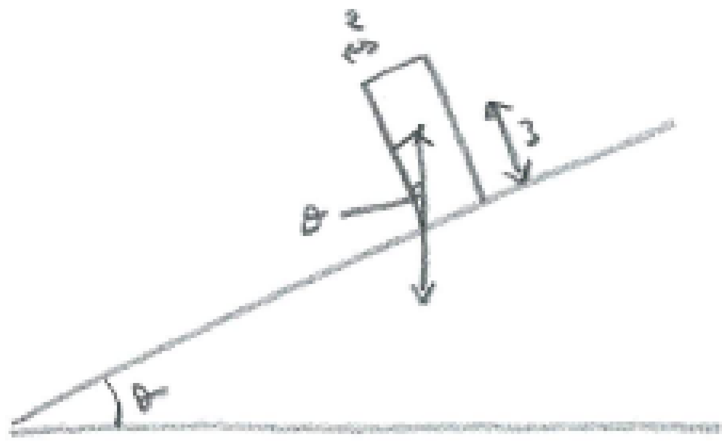

Figure 15

$tan\theta = \frac{2}{3}$

$\Rightarrow\ \theta = 33.7°\ (3sf)$