Arithmetic Series - Exercises (with Sol'ns)

(8 pages; 6/2/26)

(1) For each of the following arithmetic sequences, find an expression for a_k:

(a) in the form $a_k = p + q(k-1)$

(b) in the form $a_k = mk + c$

(c) in the form $a_k = a_{k-1} + t$; $a_1 = u$ $(k \geq 2)$

(where p, q, m, c, t & u are to be found)

(i) $4, 7, 10, 13, 16, \ldots$

(ii) $-2, -1, 0, 1, 2, \ldots$

(iii) $8, 6, 4, 2, 0, \ldots$

Solution

(i) (a) $a_k = 4 + 3(k-1)$

(b) $a_k = 3k + 1$

(c) $a_k = a_{k-1} + 3$; $a_1 = 4$ $(k \geq 2)$

(ii)(a) $a_k = -2 + (k-1)$

(b) $a_k = k - 3$

(c) $a_k = a_{k-1} + 1$; $a_1 = -2$ $(k \geq 2)$

(iii)(a) $a_k = 8 - 2(k-1)$

(b) $a_k = 10 - 2k$

(c) $a_k = a_{k-1} - 2$; $a_1 = 8$ $(k \geq 2)$

(2) If I pay £50 into a bank account, then £60 a year later, followed by £70 the following year, and so on, increasing by £10 each year, how long will it take for the amount in the bank account to reach £1000?

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Solution

Consider $\frac{n}{2}[(2(50) + 10(n-1)] = 1000$

$\Rightarrow n(90 + 10n) = 2000$

$\Rightarrow n^2 + 9n - 200 = 0$

$\Rightarrow n = \frac{-9 \pm \sqrt{81+800}}{2} = 10.3$ (as $n > 0$)

So, at the start of the 11th year (after paying in the amount due then) there will be over £1000 in the account.

(3)(i) If teams A, B, C, D & E in some sporting competition have to play each other once, how many games are there in total?

(ii) Extend this to find a formula for $1 + 2 + 3 + \cdots + n$

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Solution

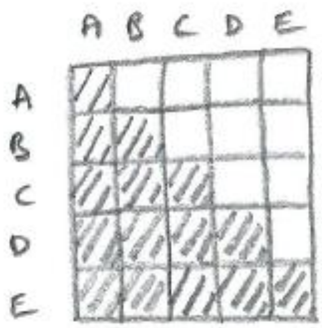
(i) AvB, AvC, AvD, AvE   4 games

BvC, BvD, BvE   3 games

CvD, CvE 2 games

DvE 1 game

Total $= 1 + 2 + 3 + 4 = 10$ games



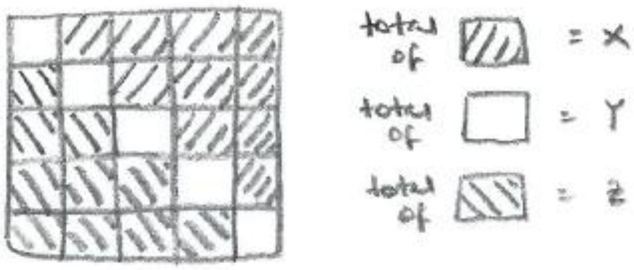
(ii) Consider the case $n = 4$

Divide the 5×5 square up into the areas X, Y & Z

Let the squares be of unit area.

Then $X = 1 + 2 + 3 + 4$,

$Y = 5$ & $Z = X$



So, for $n = 4$, $X + 5 + X = 25$

Generalising this, $2X + (n + 1) = (n + 1)^2$

$\Rightarrow 2X = (n + 1)[(n + 1) - 1] = (n + 1)n$

$\Rightarrow X = \frac{n(n+1)}{2}$

(4) For an arithmetic sequence with 1st term a and common difference d, show that the sum of the 1st n terms is $\frac{n}{2}[2a+(n-1)d]$ by starting with $\sum_{k=1}^{n}[a+(k-1)d]$

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Solution

$\sum_{k=1}^{n}[a+(k-1)d] = [(a-d)\sum_{k=1}^{n}1] + d\sum_{k=1}^{n}k$

$= (a-d)n + d\,.\frac{1}{2}\,n(n+1)$

$= \frac{n}{2}(2a-2d+dn+d) = \frac{n}{2}[2a+(n-1)d]$