Algorithms – Q8 (20/11/23)

By performing traces, or otherwise, establish what the following algorithm achieves.

Step 1: Two positive integers are entered.

Step 2: If the two numbers are equal, then output their common value. Otherwise go to Step 3.

Step 3: Divide the larger number by the smaller one (possibly with a remainder). Then go to Step 4.

Step 4: If the division from Step 3 is exact, then output the divisor [ie the number that we are dividing by]. Otherwise go to Step 5.

Step 5: If the division carried out in Step 3 is not exact, then let the divisor and the remainder be the two new numbers, and go to Step 3.

Solution

Let $b = ka + r$, where a & k are positive integers, and r is a non-negative integer, with $0 \leq r < a$.

The algorithm uses the result that $hcf(a, b) = hcf(a, r)$.

[See STEP, Pure Exercises, Integers Q7 for proof.]

The pair (a, b) is replaced with the pair (a, r), and the process is then repeated with (r, a), and so on until $r = 0$. (At each stage, $r < a$, and so $r = 0$ after a finite number of steps.