Second Order Linear Differential Equations with Constant Coefficients (17 pages; 4/3/25)

Exercise: Substitute $y = Ae^{\lambda x}$ into the eq'n $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$ and find λ

[Substitute $y = Ae^{\lambda x}$ into the eq'n $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$ and find λ]

$\lambda^2 Ae^{\lambda x} - \lambda Ae^{\lambda x} - 6Ae^{\lambda x} = 0$

$\Rightarrow \lambda^2 - \lambda - 6 = 0$

or $(\lambda + 2)(\lambda - 3) = 0$, which has roots $\lambda = -2$ & 3.

The general solution of $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$ is then

$y = Ae^{-2x} + Be^{3x}$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = x^2$$

Trial function for particular integral?

$[\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = x^2]$

Trial function: $y = ax^2 + bx + c$

Find a, b & c

$[\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = x^2$; trial function: $y = ax^2 + bx + c]$

$2a - (2ax + b) - 6(ax^2 + bx + c) = x^2$

Equating coeffs:

$x^2: -6a = 1$; $x: -2a - 6b = 0$; x^0: $2a - b - 6c = 0$

leading to $a = -\frac{1}{6}, b = \frac{1}{18}$ $, c = -\frac{7}{108}$

General solution:

$y = Ae^{-2x} + Be^{3x} - \frac{x^2}{6} + \frac{x}{18} - \frac{7}{108}$

Find A & B, given the boundary conditions:

$y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$

[$y = Ae^{-2x} + Be^{3x} - \frac{x^2}{6} + \frac{x}{18} - \frac{7}{108}$

$y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$]

$1 = A + B - \frac{7}{108}$

and $2 = -2A + 3B + \frac{1}{18}$

leading to $A = \frac{1}{4}$ and $B = \frac{22}{27}$

Particular solution:

$y = \frac{1}{4}e^{-2x} + \frac{22}{27}e^{3x} - \frac{x^2}{6} + \frac{x}{18} - \frac{7}{108}$

Exercise: Find the Complementary Function for the eq'n

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = cos2x$$

$[\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = cos2x\ ]$

Aux. eq'n: $\lambda^2 - 6\lambda + 9 = 0$ or $(\lambda - 3)^2 = 0 \Rightarrow \lambda = 3$ (repeated)

CF: $y = (A + Bx)e^{3x}$

Trial function for the particular integral?

$[\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = cos2x$; CF: $y = (A + Bx)e^{3x}$]

Trial function: $y = asin2x + bcos2x$

Find a & b

[$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = cos2x$; CF: $y = (A + Bx)e^{3x}$

Trial function: $y = asin2x + bcos2x$]

$(-4asin2x - 4bcos2x) - 6(2acos2x - 2bsin2x)$

$+9(asin2x + bcos2x) = cos2x$

Equating coefficients of $sin2x$ and $cos2x$:

$-4a + 12b + 9a = 0$ and $-4b - 12a + 9b = 1$

leading to $a = -\frac{12}{169}$ and $b = \frac{5}{169}$

General solution: $y = (A + Bx)e^{3x} - \frac{12sin2x}{169} + \frac{5cos2x}{169}$

Exercise: Find the Complementary function for the eq’n

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = e^{3x}$$

$[\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = e^{3x}\ ]$

Aux. eq'n: $\lambda^2 - 2\lambda + 5 = 0 \Rightarrow \lambda = 1 \pm 2i$

CF: $y = e^x(Ae^{2ix} + Be^{-2ix})$

$= e^x\{Acos2x + iAsin2x + Bcos(-2x) + iBsin(-2x)\}$

$= e^x\{(A + B)cos2x + i(A - B)sin2x\}$

$= e^x(Ccos2x + Dsin2x)$

[A and B have to be complex conjugates]

[or $Ee^x sin(2x + \alpha)$ etc]

Exercise: Find the general solution of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = e^{3x}$

[$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = e^{3x}$; CF: $y = e^x(Ccos2x + Dsin2x)$]

Trial function: ae^{3x}

Substituting into (*) gives $9ae^{3x} - 2(3ae^{3x}) + 5ae^{3x} = e^{3x}$,

so that $8a = 1$ and $a = \frac{1}{8}$.

General sol'n: $y = e^x(Ccos2x + Dsin2x) + \frac{e^{3x}}{8}$

Exercise: Find the general solution of $\frac{d^2y}{dx^2} + 9y = sin3x$

$[\frac{d^2y}{dx^2} + 9y = sin3x]$

Aux. eq'n: $\lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i$

CF: $y = Asin3x + Bcos3x$

Exercise : Find the general sol'n

$[\frac{d^2y}{dx^2} + 9y = sin3x$;CF: $y = Asin3x + Bcos3x]$

Trial function: $x(asin3x + bcos3x)$

$\frac{dy}{dx} = (asin3x + bcos3x) + x(3acos3x - 3bsin3x)$

and $\frac{d^2y}{dx^2} = 3acos3x - 3bsin3x + (3acos3x - 3bsin3x)$

$+x(-9asin3x - 9bcos3x)$

$= 6acos3x - 6bsin3x - 9x(asin3x + bcos3x)$

Then, substituting into (*);

$6acos3x - 6bsin3x - 9x(asin3x + bcos3x)$

$+9x(asin3x + bcos3x) = sin3x$

so that $6acos3x - 6bsin3x = sin3x$

Equating coeffs of $sin3x$ and $cos3x$:

$a = 0$ and $b = -\frac{1}{6}$

General sol'n: $y = Asin3x + Bcos3x - \frac{1}{6}xcos3x$

Trial functions for the particular integral

RHS of differential equation	Trial function
$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ ($a_n \neq 0$, but a_i may be zero for $i \neq n$)	$b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ (if $a_i = 0$, we don't set $b_i = 0$)
ae^{px}	be^{px}
$a_1 sinpx + a_2 cospx$ (where one of the a_i may be zero)	$b_1 sinpx + b_2 cospx$ (if $a_i = 0$, we don't set $b_i = 0$)

Notes

(i) If the RHS is contained within the CF, include a factor of x in the trial function.

(ii) If the RHS is $ae^{px}$and the CF is of the form $(A + Bx)e^{px}$, let the trial function be Cx^2e^{px}.

(iii) For a combination of functions on the RHS, the trial function will be the corresponding combination of trial functions; eg for

$x^2 + cos3x - 2e^{2x}$ it would be

$ax^2 + bx + c + csin3x + dcos3x + fe^{2x}$

(and equate coeffs of $x^2, x, x^0, sin3x, cos3x$ & e^{2x})