Vectors - Summary of results and methods (7 pages; 4/8/18)

(1) Vector product

$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & a_1 & b_1 \\ \underline{j} & a_2 & b_2 \\ \underline{k} & a_3 & b_3 \end{vmatrix}$ (or the transpose of this)

(2) Equation of a line

(a) $\underline{r} = \underline{a} + \lambda \underline{d}$

(b) $\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a})$

(c) $\underline{r} = (1 - \lambda)\underline{a} + \lambda \underline{b}$

(d) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} a_1 + \lambda d_1 \\ a_2 + \lambda d_2 \\ a_3 + \lambda d_3 \end{pmatrix}$

(e) $\lambda = \frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}$

(f) $(\underline{r} - \underline{a}) \times \underline{d} = \underline{0}$ or $\underline{r} \times \underline{d} = \underline{a} \times \underline{d}$

(3) Equation of a plane

(a) $\underline{r}.\underline{n} = a.\underline{n} = d$ ('scalar product' form)

(normal $\underline{n}$, passing through the point with position vector $\underline{a}$)

(b) $\underline{r}.\hat{\underline{n}} = d'$, where $\hat{\underline{n}} = \frac{n}{|\underline{n}|}$ and $d' = \frac{d}{|\underline{n}|}$ is the shortest distance from the plane to the Origin

(c) $n_1x + n_2y + n_3z = d$ (cartesian form)

(derived from the 'scalar product' form)

(d) $\underline{r} = \underline{a} + \lambda\underline{d} + \mu\underline{e}$ (parametric form)

Notes

(i) This can be converted to the 'scalar product' form by taking

$\underline{n} = \underline{d} \times \underline{e}$

(Alternatively, obtain the cartesian form by eliminating λ and μ from $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} + \mu \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$)

(ii) $\underline{d}$ and $\underline{e}$ can be obtained from the points $\underline{a}$, $\underline{b}$ & $\underline{c}$ in the plane (eg $\underline{d} = \underline{b} - \underline{a}$ and $\underline{e} = \underline{c} - \underline{a}$)

(iii) To convert from cartesian to parametric form, let $x = s$ and $y = t$, to find z in terms of s and t, and giving

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ ? \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ ? \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ ? \end{pmatrix}$$

(4) Angle between a line (with direction $\underline{d}$) and a plane (with normal $\underline{n}$)

Find the acute angle between $\underline{n}$ and $\underline{d}$, and subtract it from 90°.

(5) The angle between two planes is the acute angle between the normals of the planes.

(6) Vector perpendicular to a given (2D) vector:

$\binom{-b}{a}$ is perpendicular to $\binom{a}{b}$

(7) Vector perpendicular to the (3D) vectors $\underline{a}$ and $\underline{b}$

Method 1

$\underline{a} \times \underline{b}$

Method 2

Let $\underline{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ be the required vector.

Then eliminate two of d_1, d_2 & d_3 from $\underline{d}.\underline{a} = 0$ and $\underline{d}.\underline{b} = 0$ (*)

to give a direction vector in terms of parameter d_1, d_2 or d_3.

eg $\begin{pmatrix} d_1 \\ 2d_1 \\ 3d_1 \end{pmatrix}$, which is equivalent to the direction vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(8) Intersection of the two lines

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$:

eliminate λ and μ from $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$

Note: If no solution exists (ie if the equations are not consistent), then the lines are skew.

(9) Point of intersection of the line $\underline{r} = \underline{a} + \lambda\underline{d}$ and the plane $\underline{r}.\underline{n} = b.\underline{n} \Rightarrow (\underline{a} + \lambda\underline{d}).\underline{n} = b.\underline{n}$, giving a value for λ, and hence the required point on the line.

(10) Line of intersection of two planes

Method 1

Substitute $x = \lambda$ into the cartesian equations of the two planes, and find y and z in terms of λ, to give

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ ? \\ ? \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ ? \\ ? \end{pmatrix}$$

Method 2

Find two points, $\underline{a}$ and $\underline{b}$, that lie on both of the planes (and hence on the line); eg by setting $x = 0$ (for one point) and $y = 0$ (for another).

The equation of the intersecting line is then $\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a})$

Method 3

Find a point that lies on both of the planes; then for the direction of the line, take the vector product of the normals of the two planes (as the line will be perpendicular to both of these).

(11) Shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$

Method 1

Obtain the unit normal vector $\underline{\hat{n}} = \frac{\underline{n}}{|\underline{n}|}$

and rewrite $\underline{r}.\underline{n} = d$ as $\underline{r}.\underline{\hat{n}} = d'$, where $d' = \frac{d}{|\underline{n}|}$

Then consider the line $\underline{r} = \underline{p} + \lambda\underline{\hat{n}}$, and the point where this meets the plane; ie where $(\underline{p} + \lambda\underline{\hat{n}}).\underline{\hat{n}} = d'$

The value of λ obtained from this eq'n gives the required distance: $|\lambda|$.

Method 2

Create the equation of the plane passing through $\underline{p}$, parallel to the plane $\underline{r}.\underline{\hat{n}} = d'$, to give $\underline{r}.\underline{\hat{n}} = e'$

Then the required distance is $|d' - e'|$

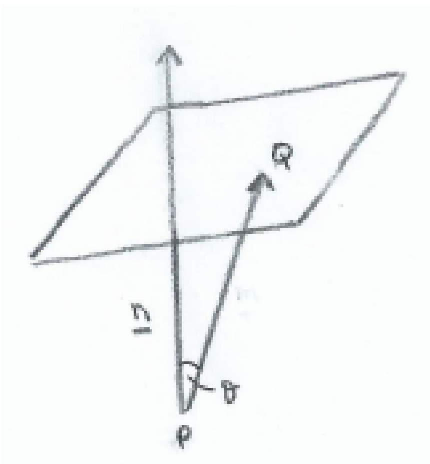
Note

This is how the standard formula $\frac{|n_1p_1+n_2p_2+n_3p_3-d|}{\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}}$ is derived:

$d' = \frac{d}{\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}}$ and $e' = \underline{p}.\underline{\hat{n}} = \frac{n_1p_1+n_2p_2+n_3p_3}{\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}}$

Method 3

Find any point Q in the plane (eg by setting $x = y = 0$ in the cartesian form).



The required distance will then be the projection of $\overrightarrow{PQ}$ onto $\underline{n}$ (the normal to the plane); namely $\frac{|\overrightarrow{PQ}.\underline{n}|}{|\underline{n}|}$

(12) Distance between two parallel planes

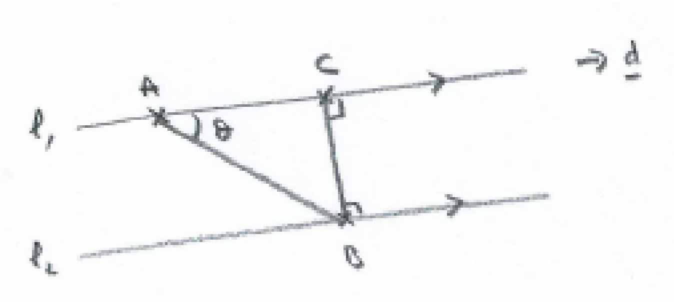
As for the shortest distance from a point to a plane, if the two planes are written in the form $\underline{r}.\hat{\underline{n}} = d'$ and $\underline{r}.\hat{\underline{n}} = e'$

(where $\hat{\underline{n}} = \frac{\underline{n}}{|\underline{n}|}$ is the unit normal vector, and $d' = \frac{d}{|\underline{n}|}$ (and similarly for e')),

then the required distance is $|d' - e'|$

(13) Distance between parallel lines / shortest distance from a point to a line

If A and B are given points on the two lines, and $\underline{d}$ is the common direction vector:



Method 1

Let C be the point on l_1 with parameter k, so that $\underline{c} = \underline{a} + k\underline{d}$ (*)

Then we require $\underline{d}.(\underline{c} - \underline{b}) = 0$

Solving this equation for k and substituting for k in (*) gives $\underline{c}$, and the distance between the two lines is then $|\underline{c} - \underline{b}|$.

Method 2

Having obtained the general point, C on l_1 in Method 1, we can minimise the distance BC by finding the stationary point of BC^2 (ie where $\frac{d}{dk}(BC^2) = 0$)

Method 3

As $BC = ABsin\theta,\ BC = \frac{|\overrightarrow{AB} \times \underline{d}|}{|\underline{d}|}$

Method 4 (2D lines)

The line equivalent of the formula $\frac{|n_1b_1+n_2b_2+n_3b_3-d|}{\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}}$ for the shortest distance from a point to a plane (see above) gives $\frac{|n_1b_1+n_2b_2-d|}{\sqrt{{n_1}^2+{n_2}^2}}$ as the shortest distance from the point $B(b_1, b_2)$ to the line $n_1x + n_2y = d$

(14) Shortest distance between two skew lines

$|(\underline{c} - \underline{a}).\frac{(\underline{b} \times \underline{d})}{|\underline{b} \times \underline{d}|}|$

$(l_1\text{:}\ \underline{r} = \underline{a} + \lambda\underline{b} \quad \&\ l_2\text{:}\ \underline{r} = \underline{c} + \mu\underline{d})$

Note

Two lines in 3D will intersect if $(\underline{c} - \underline{a}).(\underline{b} \times \underline{d}) = 0$