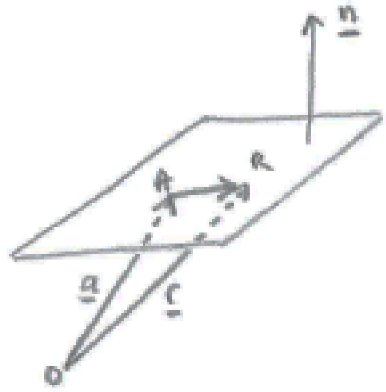
Vectors - Equation of plane (4 pages; 4/8/18)

(1) scalar product form

Let $\underline{a}$ be the position vector of a point in the plane,

and $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a general point in the plane.

Let $\underline{n}$ be a vector perpendicular to the plane (see below).



As $\underline{r} - \underline{a}$ and $\underline{n}$ are perpendicular, $(\underline{r} - \underline{a}).\underline{n} = 0$

$\Rightarrow \underline{r}.\underline{n} = \underline{a}.\underline{n} = p$ (a constant)

$\Rightarrow n_x x + n_y y + n_z z =$ p (Cartesian form)

Note: To find $\underline{n}$, given two direction vectors $\underline{d}_1$ and $\underline{d}_2$ in the plane: $\underline{n} = \underline{d}_1 \times \underline{d}_2$

Thus if $\underline{a}$, $\underline{b}$ & $\underline{c}$ are the position vectors of points in the plane, we can take $\underline{d}_1 = \underline{b} - \underline{a}$ and $\underline{d}_2 = \underline{c} - \underline{a}$, for example.

Example

If $\underline{a} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ and $\underline{n} = \begin{pmatrix} -12 \\ 11 \\ -9 \end{pmatrix}$, then

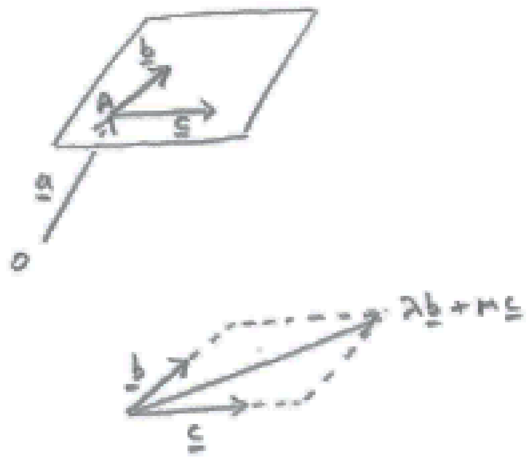
$-12x + 11y - 9z = -12(1) + 11(2) - 9(4) = -26$

(Another way of thinking of this is that, since $\underline{a}$ is a point in the plane, it is a solution of $\underline{r}.\underline{n} = p$, so that $p = \underline{a}.\underline{n}$, or $\underline{n}.\underline{a}$)

(2) Parametric form

This is an extension of the parametric form of the vector equation of a line.

Let $\underline{b}$ and $\underline{c}$ be non-zero vectors in the plane (that are not parallel to each other).



Then $\underline{r} = \underline{a} + (\lambda\underline{b} + \mu\underline{c})$

Note that $\underline{b}$ and $\underline{c}$ are direction vectors, whilst $\underline{a}$ is a position vector. $\underline{b}$ and $\underline{c}$ can of course be determined from 2 points $\underline{p}$ and $\underline{q}$ in the plane, as $\underline{p} - \underline{a}$ and $\underline{q} - \underline{a}$ (or $\underline{p} - \underline{q}$)

(3) Converting between cartesian and parametric forms

(a) to convert from cartesian to parametric form

Example

Suppose that the equation of the plane is $2x + 3y + z = 4$

Let $x = s$ and $y = t$, so that $z = 4 - 2s - 3t$ and a general point

is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ t \\ 4 - 2s - 3t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$

(b) to convert from parametric to cartesian form

Example: $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + s\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

Method 1

Create the normal vector: $\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$

$= \begin{vmatrix} \underline{i} & -1 & 2 \\ \underline{j} & 3 & 3 \\ \underline{k} & 5 & 1 \end{vmatrix} = -12\underline{i} + 11\underline{j} - 9\underline{k}$

giving $-12x + 11y - 9z = -12(1) + 11(2) - 9(4) = -26$,

as the point $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ lies in the plane.

Method 2

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + s\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$\rightarrow \quad x = 1 - s + 2t$

$y = 2 + 3s + 3t$

$z = 4 + 5s + t$

Then eliminate s and t to obtain an equation in x, y & z.