Vectors - Intersections (3 pages; 4/8/18)

(lines are 3D)

(1) Point of intersection of two lines

Note: Lines may not have a point of intersection, if the equations are not consistent; in which case they are termed 'skew'.

Example: intersection of l_1 & l_2 ,

where l_1 has equation $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

and l_2 has equation $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

Eliminate λ & μ, to give $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$

(2) Point of intersection of a line and a plane

Example: l_1 has equation $\underline{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$;

plane has equation $\underline{r}.\begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} = d$ (where d is a specific number)

Then $\left(\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\right).\begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} = d$ creates a linear equation in λ.

(It is possible that the line is either parallel to the plane or lies in the plane; in which case $\lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.\begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$ above will vanish, since the scalar product will be zero; then the remaining numbers will only be consistent if $\begin{pmatrix}2\\3\\4\end{pmatrix}$ lies in the plane; ie if $\begin{pmatrix}2\\3\\4\end{pmatrix}.\begin{pmatrix}5\\1\\-1\end{pmatrix} = d$)

(3) Line of intersection of two planes

Method 1

Planes $2x + z = 3$ & $x + y - z = 2$

Let $x = \lambda$, so that $z = 3 - 2\lambda$ and $y = 2 + (3 - 2\lambda) - \lambda = 5 - 3\lambda$

Then the equation of the line of intersection of the planes is:

$$\underline{r} = \begin{pmatrix}0\\5\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\-3\\-2\end{pmatrix}$$

Method 2

Find two points that lie on the line of intersection.

Planes $2x + z = 3$ & $x + y - z = 2$ (as above)

Let $x = 0$, so that $z = 3$ & $y - z = 2$; giving $y = 5$

So (0,5,3) lies on both planes, and hence on the line.

Similarly, let $z = 0$, so that $x = \frac{3}{2}$ and $y = \frac{1}{2}$

So $\left(\frac{3}{2}, \frac{1}{2}, 0\right)$ also lies on both planes, and hence on the line.

(Note: We could have chosen $y = 0$ instead, but it would have given us a pair of simultaneous equations to solve.)

Then the equation of the intersecting line is

$$\underline{r} = \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} \frac{3}{2} - 0 \\ \frac{1}{2} - 5 \\ 0 - 3 \end{pmatrix} \text{ or } \underline{r} = \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$$

Method 3

The required line will be perpendicular to the normal vectors of both planes. Therefore the vector product of the normal vectors to the two planes has the direction vector of the required line.

Planes $2x + z = 3$ & $x + y - z = 2$ (as above):

$$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{vmatrix} \underline{i} & 2 & 1 \\ \underline{j} & 0 & 1 \\ \underline{k} & 1 & -1 \end{vmatrix} = -\underline{i} + 3\underline{j} + 2\underline{k}$$

In order to find the equation of the line, we just need a point on it; ie a point on both planes; eg let $z = 0$, so that $x = \frac{3}{2}$ & $y = \frac{1}{2}$

and the equation of the line is $\underline{r} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$