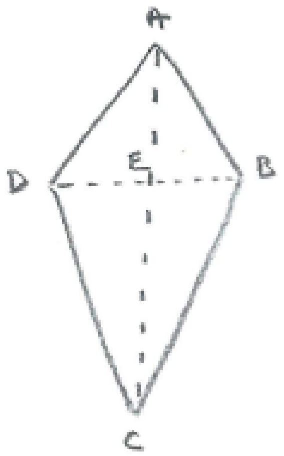
Vectors Q24 (3/7/23)

In the diagram below, ABCD is a kite. Find $\overrightarrow{OD}$ if $\overrightarrow{OA} = \begin{pmatrix} -1 \\ 4/3 \\ 7 \end{pmatrix}$,

$\overrightarrow{OB} = \begin{pmatrix} 4 \\ 4/3 \\ 2 \end{pmatrix}$ & $\overrightarrow{OC} = \begin{pmatrix} 6 \\ 16/3 \\ 2 \end{pmatrix}$



[from AEA, June 2009]

Solution

We need to take account of the special features of this case, namely that AC is perpendicular to BD (*) and bisects BD (**) (which enables D to be uniquely determined from A, B & C).

We can take an alternative route to D, in order to involve the other points, writing:

$\overrightarrow{OD} = \overrightarrow{OB} + 2\overrightarrow{BE}$ [this takes account of (**)]

To record the fact that E lies on AC, we write $\overrightarrow{BE} = \overrightarrow{BA} + \lambda\overrightarrow{AC}$

We also need to take account of (*): $\overrightarrow{BE}.\overrightarrow{AC} = 0$

Now $\overrightarrow{BA} = \begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 7 \\ 4 \\ -5 \end{pmatrix}$, so that $\overrightarrow{BE} = \begin{pmatrix} -5+7\lambda \\ 4\lambda \\ 5-5\lambda \end{pmatrix}$

$\overrightarrow{BE}.\overrightarrow{AC} = 0$ gives $7(-5+7\lambda) + 4(4\lambda) -5(5-5\lambda) = 0$

so that $90\lambda = 60$ and $\lambda = 2/3$

Hence $\overrightarrow{BE} = \begin{pmatrix} -1/3 \\ 8/3 \\ 5/3 \end{pmatrix}$

Then $\overrightarrow{OD} = \overrightarrow{OB} + 2\overrightarrow{BE} = \begin{pmatrix} 4-2/3 \\ \frac{4}{3}+16/3 \\ 2+10/3 \end{pmatrix} = \begin{pmatrix} 10/3 \\ 20/3 \\ 16/3 \end{pmatrix} = 2/3\begin{pmatrix} 5 \\ 10 \\ 18 \end{pmatrix}$