Vectors Q18 (3/7/23)

Find the distance between the lines $\frac{x+1}{1} = \frac{y+2}{2}; z = 4$ and $\frac{x+3}{1} = \frac{y-6}{2}\ ; z = 7$, leaving your answer in exact form.

Solution

Method 1

The lines are parallel.

Choose a point on one of the lines; eg $P = (-3,6,7)$ on the 2nd line.

To find the distance of this point from the 1st line:

A general point, Q on the 1st line is $\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} -1+\lambda \\ -2+2\lambda \\ 4 \end{pmatrix}$

Then $\overrightarrow{PQ} = \begin{pmatrix} -1+\lambda \\ -2+2\lambda \\ 4 \end{pmatrix} - \begin{pmatrix} -3 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 2+\lambda \\ -8+2\lambda \\ -3 \end{pmatrix}$

We want $\overrightarrow{PQ}$ to be perpendicular to the 1st line,

so that $\begin{pmatrix} 2+\lambda \\ -8+2\lambda \\ -3 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 0$

$\Rightarrow 2 + \lambda - 16 + 4\lambda = 0 \ \Rightarrow 5\lambda = 14; \lambda = \frac{14}{5}$

Then $\overrightarrow{PQ} = \begin{pmatrix} \frac{24}{5} \\ -\frac{12}{5} \\ -\frac{15}{5} \end{pmatrix} = \frac{3}{5}\begin{pmatrix} 8 \\ -4 \\ -5 \end{pmatrix}$ and the required distance is $\frac{3}{5}\sqrt{64+16+25}$

$= \boldsymbol{\frac{3\sqrt{105}}{5}}$

Method 2

Choose a point on each line; eg $R=(-1,-2,4)$ on the 1st line, and
$P=(-3,6,7)$ on the 2nd line.

Then $\overrightarrow{PR}=\begin{pmatrix}2\\-8\\-3\end{pmatrix}$ and the required distance is $\left|\frac{\begin{pmatrix}2\\-8\\-3\end{pmatrix}\times\begin{pmatrix}1\\2\\0\end{pmatrix}}{\left|\begin{pmatrix}1\\2\\0\end{pmatrix}\right|}\right|$

$$=\left|\frac{\begin{vmatrix}\underline{i} & 2 & 1\\ \underline{j} & -8 & 2\\ \underline{k} & -3 & 0\end{vmatrix}}{\sqrt{5}}\right|=\frac{1}{\sqrt{5}}\left|\begin{pmatrix}6\\-3\\12\end{pmatrix}\right|=\frac{3}{\sqrt{5}}\left|\begin{pmatrix}2\\-1\\4\end{pmatrix}\right|=\frac{3}{\sqrt{5}}\sqrt{21}=\frac{3\sqrt{105}}{5}$$