Vectors Q16 (3/7/23)

Find the shortest distance between the point $(4, -2, 3)$ and the line $\underline{r} = \begin{pmatrix} 7 \\ 5 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -6 \\ 4 \end{pmatrix}$, leaving the answer in surd form.

Solution

Method 1

Shortest distance $D = \frac{|(\underline{p}-\underline{a})\times\underline{d}|}{|\underline{d}|}$ where $\underline{p} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$, $\underline{a} = \begin{pmatrix} 7 \\ 5 \\ -1 \end{pmatrix}$ &

$\underline{d} = \begin{pmatrix} 3 \\ -6 \\ 4 \end{pmatrix}$

$$(\underline{p} - \underline{a}) \times \underline{d} = \begin{vmatrix} \underline{i} & -3 & 3 \\ \underline{j} & -7 & -6 \\ \underline{k} & 4 & 4 \end{vmatrix} = \begin{pmatrix} -4 \\ 24 \\ 39 \end{pmatrix}$$

So $D = \frac{\sqrt{(-4)^2+24^2+39^2}}{\sqrt{3^2+(-6)^2+4^2}} = \frac{\sqrt{2113}}{\sqrt{61}} = \sqrt{\frac{2113}{61}}$

Method 2

Let $\overrightarrow{OP} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$ and let M be the point on the line closest to P,

so that $\overrightarrow{OM} = \begin{pmatrix} 7 + 3\lambda \\ 5 - 6\lambda \\ -1 + 4\lambda \end{pmatrix}$, for some λ to be determined.

We require $\overrightarrow{PM}.\underline{d} = 0$,

so that $\begin{pmatrix} 7 + 3\lambda - 4 \\ 5 - 6\lambda - (-2) \\ -1 + 4\lambda - 3 \end{pmatrix} . \begin{pmatrix} 3 \\ -6 \\ 4 \end{pmatrix} = 0$

$\Rightarrow 3(3 + 3\lambda) - 6(7 - 6\lambda) + 4(-4 + 4\lambda) = 0$

$\Rightarrow -49 + 61\lambda = 0 \Rightarrow \lambda = \frac{49}{61}$

The shortest distance $D = \left|\overrightarrow{PM}\right|$

where $\overrightarrow{PM} = \begin{pmatrix} 7 + 3\lambda - 4 \\ 5 - 6\lambda - (-2) \\ -1 + 4\lambda - 3 \end{pmatrix} = \begin{pmatrix} 3 + 3\lambda \\ 7 - 6\lambda \\ -4 + 4\lambda \end{pmatrix}$

So $D^2 = (3 + \frac{147}{61})^2 + (7 - \frac{294}{61})^2 + (-4 + \frac{196}{61})^2$

$= \frac{1}{61^2}(330^2 + 133^2 + (-48)^2)$

$= \frac{128893}{61^2} = \frac{2113}{61}$

and $D = \sqrt{\frac{2113}{61}}$