Vectors Q12 (3/7/23)

(i) Find the intersection of the line $\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and the plane $3x + y + 4z = 77$

(ii) Find the shortest distance from the point $\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$ to the plane $3x + y + 4z = 77$

Solution

(i) For a point on the line, $x = 2 + 3\lambda, y = -1 + \lambda, z = 5 + 4\lambda$

Substituting into the eq'n of the plane:

$3(2 + 3\lambda) + (-1 + \lambda) + 4(5 + 4\lambda) = 77$

$\Rightarrow 26\lambda = 77 - 25 \Rightarrow \lambda = \frac{52}{26} = 2$

$\Rightarrow$ point of intersection has position vector $\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 8 \\ 1 \\ 13 \end{pmatrix}$

(ii) From (i), nearest point on the plane is $\begin{pmatrix} 8 \\ 1 \\ 13 \end{pmatrix}$,

so that shortest distance is $\sqrt{(8-2)^2 + (1-[-1])^2 + (13-5)^2}$

$= \sqrt{36 + 4 + 64} = \sqrt{104} = 2\sqrt{26}$

Alternative method

From (i), shortest distance is $|\lambda||\underline{n}|$, where λ corresponds to the point of intersection of the line and plane, and $\underline{n}$ is the normal vector for the plane; ie $2\sqrt{3^2 + 1^2 + 4^2} = 2\sqrt{26}$