Vectors - Consistency of 3 simultaneous equations

(8 pages; 4/7/16)

Consider the equations

$x + 2y + 3z = 4 \quad (1)$

$2x - y + 4z = 1 \quad (2)$

$ax + 3y - z = b \quad (3)$

These can be interpreted as 3 planes.

Suppose that $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ a & 3 & -1 \end{vmatrix} = 0$, leading to $a = -1$, so that there isn't a unique solution, and (in general) one of the following will apply:

Situations involving parallel planes

(a) the 3 planes are identical

(b) two of the planes are identical and the 3rd is parallel to the other two

(c) two of the planes are identical and the 3rd is **not** parallel to the other two: this could be classified with (f) below (the sheaf of planes), as the 3 planes have a common line

(d) the 3 planes are all parallel to each other (and no two are identical)

(e) two of the planes are parallel (and not identical), and the 3rd is not parallel to the other two

Other situations

(f) the 3 planes have a common line [sheaf of planes] ie the equations are consistent

(g) two of the planes have a common line, but the 3rd is parallel to the line (ie the direction vector of the line is parallel to a vector in the 3rd plane) [triangular prism]

ie the equations are not consistent

Summary

Consistent - infinite number of solutions (plane): (a)

Consistent - infinite number of solutions (line): (c) & (f)

Inconsistent: (b), (d), (e) & (g)

In our example, no two planes are parallel, and so either (f) or (g) applies. Suppose that we wish to find the value of b such that (f) applies and the equations are consistent.

Approach 1

We can use one equation to eliminate one of the variables, and then ensure that the remaining equations are consistent.

For example, (2) $\Rightarrow y = 2x + 4z - 1$

Then (1) & (3) give:

$x + 2(2x + 4z - 1) + 3z = 4$ & $-x + 3\,(2x + 4z - 1) - z = b$

so that $5x + 11z = 6$

and $5x + 11z = b + 3$

and hence $b = 3$

Approach 2

We can set x (for example) equal to a parameter λ, and use equations (1) & (2) to express y & z in terms of λ. Then we find the value of b such that equation (3) holds.

Thus (1) & (2) ⇒

$2y + 3z = 4 - \lambda \quad (4)$

$-y + 4z = 1 - 2\lambda \ (5) \Rightarrow -2y + 8z = 2 - 4\lambda \ \ (5')$

Adding (4) & (5):

$11z = 6 - 5\lambda \Rightarrow z = \frac{1}{11}(6 - 5\lambda)$

& (5) $\Rightarrow y = \frac{4}{11}(6 - 5\lambda) + 2\lambda - 1 = \frac{1}{11}(13 + 2\lambda)$

Then, substituting into (3):

$b = -\lambda + \frac{3}{11}(13 + 2\lambda) - \frac{1}{11}(6 - 5\lambda) = 3$

Approach 3

If (f) is to apply, then we can first of all find the equation of the line of intersection of planes (1) & (2); L, say:

There will be a point on L with a z coordinate of 0 (choosing z because of the slightly larger coefficients, 3 & 4, which will now disappear).

Then (1) & (2) become:

$x + 2y = 4$

& $2x - y = 1$

[The planes might of course both be parallel to the z-axis (ie the z-axis is parallel to a vector in the plane) and not have any z terms; in which case either x or y can be used.]

As a change from the usual methods, we could solve these by Cramer's rule:

$$x = \frac{\begin{vmatrix} 4 & 2 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}} = \frac{-6}{-5} = \frac{6}{5} \quad \text{and} \quad y = \frac{\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}} = \frac{-7}{-5} = \frac{7}{5}$$

(which can be extended to larger numbers of equations)

Thus the point $\begin{pmatrix} 6/5 \\ 7/5 \\ 0 \end{pmatrix}$ lies on L.

The direction vector of L can be found by taking the vector product of the normals to the planes (1) & (2):

$$\begin{vmatrix} \underline{i} & 1 & 2 \\ \underline{j} & 2 & -1 \\ \underline{k} & 3 & 4 \end{vmatrix} = \begin{pmatrix} 11 \\ 2 \\ -5 \end{pmatrix}$$

Thus the equation of L is $\underline{r} = \begin{pmatrix} 6/5 \\ 7/5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 11 \\ 2 \\ -5 \end{pmatrix}$,

and we can find b by substituting for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{6}{5} + 11\lambda \\ \frac{7}{5} + 2\lambda \\ -5\lambda \end{pmatrix}$

into the equation of the 3rd plane:

Thus $-\left(\frac{6}{5} + 11\lambda\right) + 3\left(\frac{7}{5} + 2\lambda\right) - (-5\lambda) = b$,

so that $3 + 0\lambda = b$;

ie if $b = 3$, then any point on L (ie any value of λ) will satisfy the original equations

[Note that the coefficient of λ (ie zero) depends only on the LHS of the original equations, and is bound to equal 0, because of the fact that

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ -1 & 3 & -1 \end{vmatrix} = 0, \text{ since this is equivalent to}$$

$$\begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} . \left[\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right] = 0$$

$$\Leftrightarrow \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} . \begin{pmatrix} 11 \\ 2 \\ -5 \end{pmatrix} = 0 \, ,$$

where the LHS expands to give the coefficient of λ]

Approach 4

If the equations had had a unique solution, we could have used Cramer's rule to find eg

$$x = \frac{\begin{vmatrix} 4 & 2 & 3 \\ 1 & -1 & 4 \\ b & 3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ -1 & 3 & -1 \end{vmatrix}}$$

As the denominator is zero, the only way in which a solution could exist is if the numerator is also zero.

[$px = q$ only has a solution when $p = 0$, if $q = 0$ also]

Then $\begin{vmatrix} 4 & 2 & 3 \\ 1 & -1 & 4 \\ b & 3 & -1 \end{vmatrix} = 0 \Rightarrow 4(-11) - (-11) + b(11) = 0$

$\Rightarrow b = 3$

An alternative justification is as follows:

Consider the point on the intersection (L) of planes (2) & (3) where $z = 0$

Then, from (1), (2) & (3):

$x + 2y = 4$

$2x - y = 1$

$-x + 3y = b$ (6)

As the original equations have no unique solution, we know that

$\alpha(x + 2y) + \beta(2x - y) + \gamma(-x + 3y) = 0$

for some α, β, γ not all zero, and holding for all x, y & z

Then, equating coefficients of x, we have:

$\alpha(1) + \beta(2) + \gamma(-1) = 0$ (7)

and equating coefficients of y:

$\alpha(2) + \beta(-1) + \gamma(3) = 0$ (7')

Also, in order for there to be a solution when $z = 0$ (from (6)):

$\alpha(4) + \beta(1) + \gamma(b) = 0$ (7'')

Then from (7), (7') & (7''), non-zero solutions for $\alpha, \beta, \gamma \Rightarrow$

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 4 & 1 & b \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 2 & 4 \\ 2 & -1 & 1 \\ -1 & 3 & b \end{vmatrix} = 0 ,$$

which is the "Cramer's rule" condition, applied to z

Notes

(i) Not all A Level examiners may recognise this method (although it has appeared in mark schemes from time to time), so it might be safest just to use it as a check (or mention "Cramer's rule").

(ii) In some cases, the determinant in the numerator could be zero, independently of b, so that it would be necessary to consider

$$\begin{vmatrix} 1 & 4 & 3 \\ 2 & 1 & 4 \\ -1 & b & -1 \end{vmatrix} \text{ and/or } \begin{vmatrix} 1 & 2 & 4 \\ 2 & -1 & 1 \\ -1 & 3 & b \end{vmatrix}$$

(so that a condition is placed on b)

Faulty method

We have to be careful to use all of the information. Thus, for example, we might add (1) and (3) to give

$5y + 2z = 4 + b \quad (8)$ &

$2x - y + 4z = 1 \quad (2)$

Then, if we set $y = \lambda$,

$(8) \Rightarrow z = \frac{1}{2}(4 + b - 5\lambda)$

and $(2) \Rightarrow x = \frac{1}{2}\big(1 + \lambda - 2(4 + b - 5\lambda)\big) = \frac{1}{2}(11\lambda - 7 - 2b)$

and it seems as though we have a solution, whatever the value of b (or, if we had been given a particular value of b, it would seem that the equations were consistent).

However, we haven't used all of the information: (1) & (3) should provide two pieces of information. (We could, for example, obtain another piece of information from (1) or (3) alone.)