Vectors - Exercises: Lines & Planes (sol'ns)

(21 pages; 13/8/19)

(1) Find a vector equation of the line that passes through the point (1,2) and is perpendicular to the line $\underline{r} = \binom{3}{4} + \lambda\binom{4}{-1}$

Solution

Method 1

The gradient of the given line is $\frac{-1}{4}$, so that the gradient of the perpendicular line is 4.

Then a vector equation of the required line is

$\underline{r} = \binom{1}{2} + \lambda\binom{1}{4}$

Method 2 (much longer, but good practice!)

Let P be the intersection of the given line (L, say) and the perpendicular line through Q(1,2). Then P can be represented as $\binom{3+4\lambda}{4-\lambda}$, for some λ to be determined.

Then, as L is perpendicular to QP, $\binom{4}{-1}.\binom{3+4\lambda-1}{4-\lambda-2} = 0$

[noting that $\binom{4}{-1}$ is the direction vector of L; not to be confused with $\binom{3+4\lambda}{4-\lambda}$, which the position vector of a point on L]

so that $4(2+4\lambda) - (2-\lambda) = 0$,

and hence $17\lambda + 6 = 0$, and $\lambda = -\frac{6}{17}$

Thus P is $\begin{pmatrix} 3+4\left(-\frac{6}{17}\right) \\ 4-\left(-\frac{6}{17}\right) \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 27 \\ 74 \end{pmatrix}$

And a vector equation of the line through P and Q is

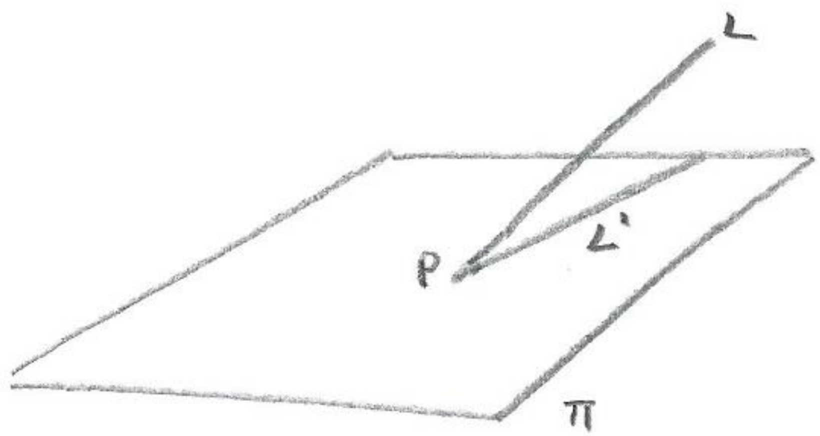
$\underline{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda[\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{17}\begin{pmatrix} 27 \\ 74 \end{pmatrix}]$

or $\underline{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{\lambda}{17}\begin{pmatrix} 17-27 \\ 34-74 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{\lambda}{17}\begin{pmatrix} -10 \\ -40 \end{pmatrix}$

or $\underline{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

(2) Given the plane Π: $3x+2y-z=6$ and the line

L: $\underline{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, let L' be the projection of L onto Π



(i) Find the point of intersection (P) of Π & L

(ii) Find the angle between Π & L

(iii) Find a vector that is parallel to Π and perpendicular to L

(iv) Find a vector equation for L'

(v) Find the angle between L and L'

Solution

(i) $3(1+2\lambda)+2(-\lambda)-(3+\lambda)=6$

$\Rightarrow 3\lambda = 6 \Rightarrow \lambda = 2$

So P is $\begin{pmatrix} 1+4 \\ -2 \\ 3+2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix}$

(ii) The angle between Π & L is the angle between L and its projection onto the plane (ie the angle between L and L'), but is most easily determined by first finding the angle between L and the normal to the plane, and subtracting this from $\frac{\pi}{2}$

If θ is the angle between L and the normal to the plane, then

$$cos\theta = \frac{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}}{\sqrt{4+1+1}\sqrt{9+4+1}} = \frac{6-2-1}{\sqrt{6}\sqrt{14}} = \frac{3}{2\sqrt{21}} = \frac{3\sqrt{21}}{2(21)} = \frac{\sqrt{21}}{14}$$

The required angle is then arcsin$\left(\frac{\sqrt{21}}{14}\right) = 19.107° = 19.1°$ (1dp)

(iii) [Note that "parallel to the plane" means parallel to a vector in the plane, and therefore perpendicular to the normal to the plane.

The required vector is also perpendicular to the plane containing L and L'.]

As the required vector is perpendicular to both the normal to the plane and L, we can use the vector product:

$$\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{vmatrix} \underline{i} & 3 & 2 \\ \underline{j} & 2 & -1 \\ \underline{k} & -1 & 1 \end{vmatrix} = \underline{i} - 5\underline{j} - 7\underline{k}$$

[A useful check is that the scalar product with the original vectors is zero. Thus $3(1) + 2(-5) + (-1)(-7) = 0$]

(iv) L' passes through P and its direction is perpendicular to both

$\underline{i} - 5\underline{j} - 7\underline{k}$ (from (iii)) and the normal to the plane.

So its direction vector is

$$\begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{vmatrix} \underline{i} & 1 & 3 \\ \underline{j} & -5 & 2 \\ \underline{k} & -7 & -1 \end{vmatrix} = 19\underline{i} - 20\underline{j} + 17\underline{k}$$

So a vector equation of L' is: $\underline{r} = \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 19 \\ -20 \\ 17 \end{pmatrix}$, from (i).

(v) If the required angle is ϕ, then

$$cos\phi = \frac{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} . \begin{pmatrix} 19 \\ -20 \\ 17 \end{pmatrix}}{\sqrt{4+1+1}\sqrt{361+400+289}} = \frac{38+20+17}{\sqrt{6}\sqrt{1050}}$$

$$= \frac{75}{2\sqrt{3}\sqrt{525}} = \frac{75}{2\sqrt{3}(5)\sqrt{21}} = \frac{15}{2(3)\sqrt{7}} = \frac{5\sqrt{7}}{14}$$

and hence $\phi = \arccos\left(\frac{5\sqrt{7}}{14}\right) = 19.107° = 19.1°$ (1dp) (as in (ii))

(3) Find the cartesian form of the plane

$$\underline{r} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

Solution

$$\underline{n} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{vmatrix} \underline{i} & 1 & 2 \\ \underline{j} & 4 & 3 \\ \underline{k} & 4 & 1 \end{vmatrix} = \begin{pmatrix} -8 \\ 7 \\ -5 \end{pmatrix}$$

$$\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}\right) . \begin{pmatrix} -8 \\ 7 \\ -5 \end{pmatrix} = 0 \Rightarrow -8x + 7(y+2) - 5(z+1) = 0$$

$\Rightarrow -8x + 7y - 5z = -9$ or $8x - 7y + 5z = 9$

Alternative version (once $\underline{n}$ has been found)

Let plane be $-8x + 7y - 5z = p$

As $\begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$ lies in the plane, $-8(0) + 7(-2) - 5(-1) = p$;

so $p = -9$ etc

Alternative method

Eliminate s & t from the 3 simultaneous equations.

(4)(i) Find the intersection of the line $\underline{r} = \underline{a} + t\underline{b}$ and the plane $\underline{r}.\underline{n} = d$

(ii) Find the intersection of the line $\underline{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and the plane $\underline{r}.\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -2$

Solution

(i) $(\underline{a} + t\underline{b}).\underline{n} = d \Rightarrow \underline{a}.\underline{n} + t\underline{b}.\underline{n} = d$

$\Rightarrow t = \frac{d - \underline{a}.\underline{n}}{\underline{b}.\underline{n}}$

$\Rightarrow \underline{r} = \underline{a} + (\frac{d - \underline{a}.\underline{n}}{\underline{b}.\underline{n}})\underline{b}$

(ii) Applying the result in (i):

$\underline{a}.\underline{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -1$

and $\underline{b}.\underline{n} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -1$

so that $\underline{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (\frac{-2-[-1]}{-1})\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

[This can be checked by representing the line and the plane in the x-y plane.]

(5) Find the line that is the reflection of the line $\frac{x-2}{3}=\frac{y}{4}=\frac{z+1}{1}$ in the plane $\boldsymbol{x-2y+z=4}$

Solution

Let the intersection of the line and the plane be P, and suppose that Q is some other point on the line. Then we can find the reflection of Q in the plane (Q' say), by dropping a perpendicular from Q onto the plane, and then the required line will pass through P and Q'.

Writing the equation of the line as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}+\lambda\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ and substituting into the equation of the plane:

$$(2+3\lambda)-2(4\lambda)+(-1+\lambda)=4 \Rightarrow -4\lambda=3;\ \lambda=-\frac{3}{4}$$

so that P is $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}-\frac{3}{4}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}=\begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix}$

Setting $\boldsymbol{\lambda=1}$ (say), we can take Q to be $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}+\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}=\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$

Now consider the perpendicular line dropped from Q onto the plane. Its direction vector is that of the normal to the plane, and so it has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Let R be the point where the perpendicular line intersects the plane. Substituting into the equation of the plane gives:

$$\boldsymbol{(5+\lambda)-2(4-2\lambda)+(\lambda)=4 \Rightarrow 6\lambda=7;\ \lambda=\frac{7}{6}}$$

So R is $\begin{pmatrix}\mathbf{5}\\\mathbf{4}\\\mathbf{0}\end{pmatrix}+\frac{7}{6}\begin{pmatrix}\mathbf{1}\\\mathbf{-2}\\\mathbf{1}\end{pmatrix}$, and Q' will be $\begin{pmatrix}\mathbf{5}\\\mathbf{4}\\\mathbf{0}\end{pmatrix}+\mathbf{2}\left(\frac{7}{6}\right)\begin{pmatrix}\mathbf{1}\\\mathbf{-2}\\\mathbf{1}\end{pmatrix}=$
$\begin{pmatrix}\frac{22}{3}\\-\frac{2}{3}\\\frac{7}{3}\end{pmatrix}$

Then , as P is $\begin{pmatrix}-\frac{1}{4}\\\mathbf{-3}\\-\frac{7}{4}\end{pmatrix}$, the equation of the reflected line will be:

$$\begin{pmatrix}\boldsymbol{x}\\\boldsymbol{y}\\\boldsymbol{z}\end{pmatrix}=\begin{pmatrix}-\frac{1}{4}\\\mathbf{-3}\\-\frac{7}{4}\end{pmatrix}+\boldsymbol{\lambda}\left[\begin{pmatrix}\frac{22}{3}\\-\frac{2}{3}\\\frac{7}{3}\end{pmatrix}-\begin{pmatrix}-\frac{1}{4}\\\mathbf{-3}\\-\frac{7}{4}\end{pmatrix}\right]=\frac{1}{12}\begin{pmatrix}\boldsymbol{-3+\lambda(88+3)}\\\boldsymbol{-36+\lambda(-8+36)}\\\boldsymbol{-21+\lambda(28+21)}\end{pmatrix}$$

$$\frac{1}{12}\begin{pmatrix}\boldsymbol{-3+91\lambda}\\\boldsymbol{-36+28\lambda}\\\boldsymbol{-21+49\lambda}\end{pmatrix}$$

or, in cartesian form: $\frac{x+\frac{3}{12}}{91}=\frac{y+\frac{36}{12}}{28}=\frac{z+\frac{21}{12}}{49}$ or $\frac{x+\frac{3}{12}}{13}=\frac{y+\frac{36}{12}}{4}=\frac{z+\frac{21}{12}}{7}$

(6)(i)(a) Find the acute angle between the line $\frac{x}{2}=\frac{y+1}{-3}=\frac{z-2}{1}$ and the plane $\boldsymbol{x+y-2z=5}$

(b) Show that the same angle is obtained if the line is written in the form

$\frac{x}{-2} = \frac{y+1}{3} = \frac{z-2}{-1}$ (ie without rearranging into the form in (a))

(ii)(a) Find the acute angle between the planes $\boldsymbol{x + 4y - 3z = 7}$ and

$\boldsymbol{x - y + 4z = 2}$

(b) Find the acute angle between the planes $\boldsymbol{x + 4y - 3z = 7}$ and

$\boldsymbol{-x + y - 4z = 2}$ (again, without rearranging the equation)

Solution

(i)(a) The angle between the line and the normal to the plane is given by

$\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \sqrt{14}\sqrt{6}\, \cos\theta$, so that $\cos\theta = \frac{-3}{\sqrt{14}\sqrt{6}} = -0.32733$

and $\theta = 109.107°$

The acute angle between these vectors is then $\mathbf{180 - 109.107 = 70.893}°$

The acute angle between the line and plane is then

$\mathbf{90 - 70.893 = 19.1}°$ (1dp)

(b) $\begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \sqrt{14}\sqrt{6}\, \cos\theta \Rightarrow \cos\theta = \frac{3}{\sqrt{14}\sqrt{6}} = 0.32733$

and $\theta = 70.893°$

As we have already found the acute angle between the line and the normal, the acute angle between the line and the plane is $\mathbf{90 - 70.893 = 19.1}°$ (1dp)

(ii) The angle between the normals to the planes is given by

$\begin{pmatrix} \mathbf{1} \\ \mathbf{4} \\ \mathbf{-3} \end{pmatrix} . \begin{pmatrix} \mathbf{1} \\ \mathbf{-1} \\ \mathbf{4} \end{pmatrix} = \sqrt{\mathbf{26}}\sqrt{\mathbf{18}}\, \boldsymbol{cos\theta}$, so that $\boldsymbol{cos\theta} = \frac{\mathbf{-15}}{\sqrt{\mathbf{26}}\sqrt{\mathbf{18}}} = \mathbf{-0.69338}$

and $\boldsymbol{\theta} = \mathbf{133.898}°$

The acute angle between the planes themselves is $\mathbf{180 - 133.898 = 46.1}°$

(ii)(b) The angle between the normals to the planes is given by

$\begin{pmatrix} \mathbf{1} \\ \mathbf{4} \\ \mathbf{-3} \end{pmatrix} . \begin{pmatrix} \mathbf{-1} \\ \mathbf{1} \\ \mathbf{-4} \end{pmatrix} = \sqrt{\mathbf{26}}\sqrt{\mathbf{18}}\, \boldsymbol{cos\theta}$, so that $\boldsymbol{cos\theta} = \frac{\mathbf{15}}{\sqrt{\mathbf{26}}\sqrt{\mathbf{18}}} = \mathbf{0.69338}$

and $\boldsymbol{\theta} = \mathbf{46.1}°$

The acute angle between the planes is also $\mathbf{46.1}°$.

(7) Find the line that is the reflection of the line $\frac{x-2}{3}=\frac{y}{4}=\frac{z+1}{1}$ in the plane $x-2y+z=4$

Solution

Let the intersection of the line and the plane be P, and suppose that Q is some other point on the line. Then we can find the reflection of Q in the plane (Q' say), by dropping a perpendicular from Q onto the plane, and then the required line will pass through P and Q'.

Writing the equation of the line as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}+\lambda\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ and substituting into the equation of the plane:

$$(2+3\lambda)-2(4\lambda)+(-1+\lambda)=4 \Rightarrow -4\lambda=3;\ \lambda=-\frac{3}{4}$$

so that P is $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}-\frac{3}{4}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}=\begin{pmatrix} -\frac{1}{4} \\ -3 \\ -\frac{7}{4} \end{pmatrix}$

Setting $\lambda=1$ (say), we can take Q to be $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}+\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}=\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$

Now consider the perpendicular line dropped from Q onto the plane. Its direction vector is that of the normal to the plane, and so it has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Let R be the point where the perpendicular line intersects the plane. Substituting into the equation of the plane gives:

$$(\mathbf{5}+\boldsymbol{\lambda})-\mathbf{2}(\mathbf{4}-\mathbf{2}\boldsymbol{\lambda})+(\boldsymbol{\lambda})=\mathbf{4} \Rightarrow \mathbf{6}\boldsymbol{\lambda}=\mathbf{7};\ \boldsymbol{\lambda}=\frac{7}{6}$$

So R is $\begin{pmatrix}\mathbf{5}\\\mathbf{4}\\\mathbf{0}\end{pmatrix}+\frac{7}{6}\begin{pmatrix}\mathbf{1}\\-\mathbf{2}\\\mathbf{1}\end{pmatrix}$, and Q' will be $\begin{pmatrix}\mathbf{5}\\\mathbf{4}\\\mathbf{0}\end{pmatrix}+\mathbf{2}\left(\frac{7}{6}\right)\begin{pmatrix}\mathbf{1}\\-\mathbf{2}\\\mathbf{1}\end{pmatrix}=$
$\begin{pmatrix}\frac{22}{3}\\-\frac{2}{3}\\\frac{7}{3}\end{pmatrix}$

Then , as P is $\begin{pmatrix}-\frac{1}{4}\\-\mathbf{3}\\-\frac{7}{4}\end{pmatrix}$, the equation of the reflected line will be:

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-\frac{1}{4}\\-\mathbf{3}\\-\frac{7}{4}\end{pmatrix}+\boldsymbol{\lambda}\left[\begin{pmatrix}\frac{22}{3}\\-\frac{2}{3}\\\frac{7}{3}\end{pmatrix}-\begin{pmatrix}-\frac{1}{4}\\-\mathbf{3}\\-\frac{7}{4}\end{pmatrix}\right]=\frac{1}{12}\begin{pmatrix}-\mathbf{3}+\boldsymbol{\lambda}(\mathbf{88}+\mathbf{3})\\-\mathbf{36}+\boldsymbol{\lambda}(-\mathbf{8}+\mathbf{36})\\-\mathbf{21}+\boldsymbol{\lambda}(\mathbf{28}+\mathbf{21})\end{pmatrix}$$

$$\frac{1}{12}\begin{pmatrix}-\mathbf{3}+\mathbf{91}\boldsymbol{\lambda}\\-\mathbf{36}+\mathbf{28}\boldsymbol{\lambda}\\-\mathbf{21}+\mathbf{49}\boldsymbol{\lambda}\end{pmatrix}$$

or, in cartesian form: $\frac{x+\frac{3}{12}}{91}=\frac{y+\frac{36}{12}}{28}=\frac{z+\frac{21}{12}}{49}$ or $\frac{x+\frac{1}{4}}{13}=\frac{y+3}{4}=\frac{z+\frac{7}{4}}{7}$

(8) Find the distance between the lines $\frac{x+1}{1}=\frac{y+2}{2}; \boldsymbol{z}=\mathbf{4}$ and $\frac{x+3}{1}=\frac{y-6}{2}$; $\boldsymbol{z}=\mathbf{7}$, leaving your answer in exact form.

Solution

Method 1

The lines are parallel.

Choose a point on one of the lines; eg $\boldsymbol{P} = (\boldsymbol{-3}, \boldsymbol{6}, \boldsymbol{7})$ on the 2nd line.

To find the distance of this point from the 1st line:

A general point, Q on the 1st line is $\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} \boldsymbol{-1+\lambda} \\ \boldsymbol{-2+2\lambda} \\ \boldsymbol{4} \end{pmatrix}$

Then $\overrightarrow{\boldsymbol{PQ}} = \begin{pmatrix} \boldsymbol{-1+\lambda} \\ \boldsymbol{-2+2\lambda} \\ \boldsymbol{4} \end{pmatrix} - \begin{pmatrix} \boldsymbol{-3} \\ \boldsymbol{6} \\ \boldsymbol{7} \end{pmatrix} = \begin{pmatrix} \boldsymbol{2+\lambda} \\ \boldsymbol{-8+2\lambda} \\ \boldsymbol{-3} \end{pmatrix}$

We want $\overrightarrow{\boldsymbol{PQ}}$ to be perpendicular to the 1st line,

so that $\begin{pmatrix} \boldsymbol{2+\lambda} \\ \boldsymbol{-8+2\lambda} \\ \boldsymbol{-3} \end{pmatrix} . \begin{pmatrix} \boldsymbol{1} \\ \boldsymbol{2} \\ \boldsymbol{0} \end{pmatrix} = \boldsymbol{0}$

$\Rightarrow \boldsymbol{2 + \lambda - 16 + 4\lambda = 0} \quad \Rightarrow \boldsymbol{5\lambda = 14}; \boldsymbol{\lambda = \frac{14}{5}}$

Then $\overrightarrow{\boldsymbol{PQ}} = \begin{pmatrix} \boldsymbol{\frac{24}{5}} \\ \boldsymbol{-\frac{12}{5}} \\ \boldsymbol{-\frac{15}{5}} \end{pmatrix} = \boldsymbol{\frac{3}{5}} \begin{pmatrix} \boldsymbol{8} \\ \boldsymbol{-4} \\ \boldsymbol{-5} \end{pmatrix}$ and the required distance is $\boldsymbol{\frac{3}{5}\sqrt{64+16+25}}$

$= \boldsymbol{\frac{3\sqrt{105}}{5}}$

Method 2

Choose a point on each line; eg $\boldsymbol{R} = (\boldsymbol{-1}, \boldsymbol{-2}, \boldsymbol{4})$ on the 1st line, and
$\boldsymbol{P} = (\boldsymbol{-3}, \boldsymbol{6}, \boldsymbol{7})$ on the 2nd line.

Then $\overrightarrow{\boldsymbol{PR}} = \begin{pmatrix} \mathbf{2} \\ \mathbf{-8} \\ \mathbf{-3} \end{pmatrix}$ and the required distance is $\left| \dfrac{\begin{pmatrix} 2 \\ -8 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right|} \right|$

$$= \left| \frac{\begin{vmatrix} \underline{i} & 2 & 1 \\ \underline{j} & -8 & 2 \\ \underline{k} & -3 & 0 \end{vmatrix}}{\sqrt{5}} \right| = \frac{1}{\sqrt{5}} \left| \begin{pmatrix} 6 \\ -3 \\ 12 \end{pmatrix} \right| = \frac{3}{\sqrt{5}} \left| \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \right| = \frac{3}{\sqrt{5}} \sqrt{21} = \frac{3\sqrt{105}}{5}$$

(9)(i) Show the lines $\frac{x-1}{2} = \frac{y+3}{5} = \frac{z-2}{3}$ and $\frac{x}{1} = \frac{y-4}{2} = \frac{z+1}{2}$ are skew.

(ii) Find the shortest distance between the lines and identify the points on the lines at which this shortest distance occurs.

Solution

(i) The lines can be rewritten in parametric form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda \\ -3 + 5\lambda \\ 2 + 3\lambda \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mu \\ 4 + 2\mu \\ -1 + 2\mu \end{pmatrix}$$

A point of intersection would then satisfy

$\mathbf{1 + 2\lambda = \mu}$ (1)
$\mathbf{-3 + 5\lambda = 4 + 2\mu}$ (2)
$\mathbf{2 + 3\lambda = -1 + 2\mu}$ (3)

Substituting from (1) into (2) & (3) gives:

$\mathbf{-3 + 5\lambda = 4 + 2(1 + 2\lambda)}$ or $\mathbf{-9 = -\lambda}$, so that $\mathbf{\lambda = 9}$

and $\mathbf{2 + 3\lambda = -1 + 2(1 + 2\lambda)}$ or $\mathbf{1 = \lambda}$, and so there is no point of intersection.

Also, the direction vectors of the lines are not parallel, and so the lines are skew.

(ii) [There are various methods for finding the shortest distance, but not all of them find the points on the lines where the shortest distance occurs. The first method given below is relatively straightforward, and doesn't involve the vector product.]

Method 1

From (i), general points on the two lines are

$$\overrightarrow{\boldsymbol{OX}} = \begin{pmatrix} \mathbf{1+2\lambda} \\ \mathbf{-3+5\lambda} \\ \mathbf{2+3\lambda} \end{pmatrix} \text{ and } \overrightarrow{\boldsymbol{OY}} = \begin{pmatrix} \boldsymbol{\mu} \\ \mathbf{4+2\mu} \\ \mathbf{-1+2\mu} \end{pmatrix}$$

At the closest approach of the two lines, $\overrightarrow{\boldsymbol{XY}}$ will be perpendicular to both lines, so that

$$\overrightarrow{\boldsymbol{XY}}.\begin{pmatrix} \mathbf{2} \\ \mathbf{5} \\ \mathbf{3} \end{pmatrix} = \mathbf{0} \text{ and } \overrightarrow{\boldsymbol{XY}}.\begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{2} \end{pmatrix} = \mathbf{0}, \text{ so that}$$

$$\begin{pmatrix} \boldsymbol{\mu} - (\mathbf{1+2\lambda}) \\ \mathbf{4+2\mu} - (\mathbf{-3+5\lambda}) \\ \mathbf{-1+2\mu} - (\mathbf{2+3\lambda}) \end{pmatrix}.\begin{pmatrix} \mathbf{2} \\ \mathbf{5} \\ \mathbf{3} \end{pmatrix} = \mathbf{0} \text{ and}$$
$$\begin{pmatrix} \boldsymbol{\mu} - (\mathbf{1+2\lambda}) \\ \mathbf{4+2\mu} - (\mathbf{-3+5\lambda}) \\ \mathbf{-1+2\mu} - (\mathbf{2+3\lambda}) \end{pmatrix}.\begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{2} \end{pmatrix} = \mathbf{0},$$

giving $(\mathbf{2\mu - 2 - 4\lambda}) + (\mathbf{35 + 10\mu - 25\lambda}) + (\mathbf{-9 + 6\mu - 9\lambda}) = \mathbf{0}$
or $\mathbf{18\mu - 38\lambda = -24}$; ie $\mathbf{9\mu - 19\lambda = -12}$ (1)

and $(\boldsymbol{\mu} - \mathbf{1} - \mathbf{2\lambda}) + (\mathbf{14 + 4\mu - 10\lambda}) + (\mathbf{-6 + 4\mu - 6\lambda}) = \mathbf{0}$

$\boldsymbol{9\mu - 18\lambda = -7}$ (2)

Then $(\mathbf{1}) - (\mathbf{2}) \Rightarrow \boldsymbol{-\lambda = -5}$, so that $\boldsymbol{\lambda = 5}$ and, from (2),

$\boldsymbol{\mu = \frac{1}{9}(18(5) - 7) = \frac{83}{9}}$

So $\overrightarrow{\boldsymbol{OX}} = \begin{pmatrix} \mathbf{11} \\ \mathbf{22} \\ \mathbf{17} \end{pmatrix}$ and $\overrightarrow{\boldsymbol{OY}} = \frac{\mathbf{1}}{\mathbf{9}}\begin{pmatrix} \mathbf{83} \\ \mathbf{202} \\ \mathbf{157} \end{pmatrix}$

and $\overrightarrow{\boldsymbol{XY}} = \frac{\mathbf{1}}{\mathbf{9}}\begin{pmatrix} \mathbf{83 - 99} \\ \mathbf{202 - 198} \\ \mathbf{157 - 153} \end{pmatrix} = \frac{\mathbf{1}}{\mathbf{9}}\begin{pmatrix} \mathbf{-16} \\ \mathbf{4} \\ \mathbf{4} \end{pmatrix} = \frac{\mathbf{4}}{\mathbf{9}}\begin{pmatrix} \mathbf{-4} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix}$,

so that $\left|\overrightarrow{\boldsymbol{XY}}\right| = \frac{\mathbf{4}}{\mathbf{9}}\sqrt{\mathbf{16 + 1 + 1}} = \frac{\mathbf{4\sqrt{18}}}{\mathbf{9}} = \frac{\mathbf{4\sqrt{2}}}{\mathbf{3}}$

Method 2 (using the vector product)

If $\underline{\hat{\boldsymbol{n}}}$ is a unit vector perpendicular to both lines, then we need $\overrightarrow{\boldsymbol{OX}}$ and $\overrightarrow{\boldsymbol{OY}}$ such that $\overrightarrow{\boldsymbol{OX}} + \boldsymbol{d}\underline{\hat{\boldsymbol{n}}} = \overrightarrow{\boldsymbol{OY}}$, and the shortest distance will then be $|\boldsymbol{d}|$.

A vector perpendicular to both lines is $\begin{pmatrix} \mathbf{2} \\ \mathbf{5} \\ \mathbf{3} \end{pmatrix} \times \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{2} \end{pmatrix} = \begin{vmatrix} \underline{\boldsymbol{i}} & \mathbf{2} & \mathbf{1} \\ \underline{\boldsymbol{j}} & \mathbf{5} & \mathbf{2} \\ \underline{\boldsymbol{k}} & \mathbf{3} & \mathbf{2} \end{vmatrix}$

$= \begin{pmatrix} \mathbf{4} \\ \mathbf{-1} \\ \mathbf{-1} \end{pmatrix}$, so that $\underline{\hat{\boldsymbol{n}}} = \frac{\mathbf{1}}{\sqrt{\mathbf{18}}}\begin{pmatrix} \mathbf{4} \\ \mathbf{-1} \\ \mathbf{-1} \end{pmatrix}$

Then $\overrightarrow{\boldsymbol{OX}} + \boldsymbol{d}\underline{\hat{\boldsymbol{n}}} = \overrightarrow{\boldsymbol{OY}}$ gives $\begin{pmatrix} \mathbf{1 + 2}\boldsymbol{\lambda} \\ \mathbf{-3 + 5}\boldsymbol{\lambda} \\ \mathbf{2 + 3}\boldsymbol{\lambda} \end{pmatrix} + \boldsymbol{D}\begin{pmatrix} \mathbf{4} \\ \mathbf{-1} \\ \mathbf{-1} \end{pmatrix} =$

$\begin{pmatrix} \boldsymbol{\mu} \\ \mathbf{4 + 2}\boldsymbol{\mu} \\ \mathbf{-1 + 2}\boldsymbol{\mu} \end{pmatrix}$, where $\boldsymbol{D} = \frac{\boldsymbol{d}}{\sqrt{18}}$,

so that $\boldsymbol{2\lambda + 4D - \mu = -1}$ (1)
$\boldsymbol{5\lambda - D - 2\mu = 7}$ (2)
$3\boldsymbol{\lambda - D - 2\mu = -3}$ (3)

Then $(\boldsymbol{2}) - (\boldsymbol{3}) \Rightarrow \boldsymbol{2\lambda = 10}$, so that $\boldsymbol{\lambda = 5}$

and (1) & (2) become $\boldsymbol{4D - \mu = -11}$ (4) and $\boldsymbol{-D - 2\mu = -18}$ (5)

Then $\boldsymbol{2(4) - (5) \Rightarrow 9D = -4}$, so that $|\boldsymbol{d}| = \sqrt{\boldsymbol{18}}|\boldsymbol{D}| = \frac{\boldsymbol{4\sqrt{18}}}{\boldsymbol{9}} = \frac{\boldsymbol{4\sqrt{2}}}{\boldsymbol{3}}$

and, from (1), $\boldsymbol{\mu = 10 - \frac{16}{9} + 1 = \frac{83}{9}}$

and $\overrightarrow{\boldsymbol{OX}}$ and $\overrightarrow{\boldsymbol{OY}}$ can then be found, as in (i).

(10) Given that $A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $B = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $D = \begin{pmatrix} p \\ 4 \\ -4 \end{pmatrix}$

(i) Write down the equations of the lines AB and CD (both extended)

(ii) Find $\overrightarrow{AB} \times \overrightarrow{CD}$

(iii) For what value of p are the lines AB and CD parallel? (2 methods)

Solution

(i) Write down the equations of the lines AB and CD (both extended)

$$\underline{r_{AB}} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-4-1\\3-2\\1-3\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-5\\1\\-2\end{pmatrix}$$

and $\underline{r_{CD}} = \begin{pmatrix}0\\1\\2\end{pmatrix} + \mu\begin{pmatrix}p\\4-1\\-4-2\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix} + \mu\begin{pmatrix}p\\3\\-6\end{pmatrix}$

Note

or eg $\underline{r_{AB}} = \begin{pmatrix}-4\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}5\\-1\\2\end{pmatrix}$

(ii) Find $\overrightarrow{AB} \times \overrightarrow{CD}$

$$\begin{pmatrix}\underline{i} & -5 & p\\ \underline{j} & 1 & 3\\ \underline{k} & -2 & -6\end{pmatrix} = -(30+2p)\underline{j} - (15+p)\underline{k} = -(15+p)(2\underline{j}+\underline{k})$$

(iii) For what value of p are the lines AB and CD parallel? (2 methods)

Method 1: Direction vectors $\begin{pmatrix}-5\\1\\-2\end{pmatrix}$ and $\begin{pmatrix}p\\3\\-6\end{pmatrix}$ need to be parallel; hence $p = -15$

Method 2: $\overrightarrow{AB} \times \overrightarrow{CD}$ must be zero

Hence $15 + p = 0$

(11) Find the plane containing the points $(2,-1,4), (-3,4,2)$ and $(1,0,5)$, in Cartesian form

Solution

Method 1

In parametric form, it is:

$$\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda\left[\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}\right] + \mu[\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}]$$

or $\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} -5 \\ 5 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$x = 2 - 5\lambda - \mu$ (1)

$y = -1 + 5\lambda + \mu$ (2)

$z = 4 - 2\lambda + \mu$ (3)

Eliminating λ & μ: $(1) + (2) \Rightarrow x + y = 1$

Method 2

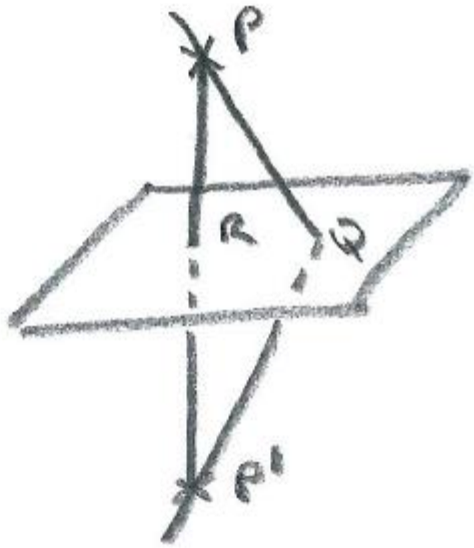
The normal to the plane is perpendicular to both

$\begin{pmatrix} -5 \\ 5 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$; eg $\begin{pmatrix} -5 \\ 5 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{vmatrix} \underline{i} & -5 & -1 \\ \underline{j} & 5 & 1 \\ \underline{k} & -2 & 1 \end{vmatrix}$

$= \begin{pmatrix} 7 \\ 7 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, so that eq'n is $\underline{r}.\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\\4\end{pmatrix}.\begin{pmatrix}1\\1\\0\end{pmatrix}$ or $x + y = 1$

(12) Find the reflection of the line $\frac{x-2}{3} = \frac{y+4}{1}; z = 3$ in the plane $y = 4$

Solution



Let P be $\begin{pmatrix}2\\-4\\3\end{pmatrix}$, say.

Q is intersection of the line and plane :

Line is $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\-4\\3\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\0\end{pmatrix}$

Substituting into the eq'n of the plane: $-4 + \lambda = 4 \Rightarrow \lambda = 8$

So Q is $\begin{pmatrix}2\\-4\\3\end{pmatrix} + 8\begin{pmatrix}3\\1\\0\end{pmatrix} = \begin{pmatrix}26\\4\\3\end{pmatrix}$

Line PR is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

R is intersection of PR and the plane:

$-4 + \mu = 4 \Rightarrow \mu = 8$

So P' is $\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} + 2(8) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix}$

Eq'n of P'Q is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix} + \theta [\begin{pmatrix} 26 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix}]$

ie $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix} + \theta \begin{pmatrix} 24 \\ -8 \\ 0 \end{pmatrix}$, or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 3 \end{pmatrix} + \theta' \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

or $\frac{x-2}{3} = \frac{y-12}{-1}; z = 3$