Useful Results - Pure (6 pages; 22/1/17)

(1) Powers of Sines and Cosines

[To deduce $sin^n\theta$ from $cos^n\theta$:

(a) write $sin^n\theta = cos^n(90-\theta)$ etc, or

(b) employ the following pattern:

(i) the coefficients are the same as for $cos^n\theta$, apart from signs

(ii) the 1st term within the brackets is

$sin\theta$; $-cos2\theta$; $-sin3\theta$; $cos4\theta$; $sin5\theta$ etc

memory aid: integrate to get to the next one (without applying the Chain rule!)

(iii) the signs of the terms within the brackets alternate]

$$cos^3\theta = \frac{1}{4}(cos3\theta + 3cos\theta)$$

$$sin^3\theta = \frac{1}{4}(-sin3\theta + 3sin\theta)$$

$$cos^4\theta = \frac{1}{8}(cos4\theta + 4cos2\theta + 3)$$

$$sin^4\theta = \frac{1}{8}(cos4\theta - 4cos2\theta + 3)$$

$$cos^5\theta = \frac{1}{16}(cos5\theta + 5cos3\theta + 10cos\theta)$$

$$sin^5\theta = \frac{1}{16}(sin5\theta - 5sin3\theta + 10sin\theta)$$

$$cos^6\theta = \frac{1}{32}(cos6\theta + 6cos4\theta + 15cos2\theta + 10)$$

$$sin^6\theta = \frac{1}{32}(-cos6\theta + 6cos4\theta - 15cos2\theta + 10)$$

$$cos^7\theta = \frac{1}{64}(cos7\theta + 7cos5\theta + 21cos3\theta + 35cos\theta)$$

$$sin^7\theta = \frac{1}{64}(-sin7\theta + 7sin5\theta - 21sin3\theta + 35sin\theta)$$

(2) $\boldsymbol{cos(n\theta), sin(n\theta)}$

[sin($2m\theta$) can't be expressed in terms of powers of $sin\theta$]

$cos3\theta = 4cos^3\theta - 3cos\theta$

$sin3\theta = -4sin^3\theta + 3sin\theta$

$cos4\theta = 8cos^4\theta - 8cos^2\theta + 1$

$cos5\theta = 16cos^5\theta - 20cos^3\theta + 5cos\theta$

$sin5\theta = 16sin^5\theta - 20sin^3\theta + 5sin\theta$

$cos6\theta = 32cos^6\theta - 48cos^4\theta + 18cos^2\theta - 1$

$cos7\theta = 64cos^7\theta - 112cos^5\theta + 56cos^3\theta - 7\cos\theta$

$sin7\theta = -64sin^7\theta + 112sin^5\theta - 56sin^3\theta + 7sin\theta$

(3) (i) For odd and even n:

$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

(ii) For odd n only:

$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1})$

(4) Tangents and normals to conics

(i) Parabola $y^2 = 4ax$ at $(at^2, 2at)$

tangent: $y = \frac{1}{t}x + at$

normal: $y = -tx + 2at + at^3$

(ii) Rectangular hyperbola $xy = c^2$

tangent: $y = -\frac{1}{t^2}x + \frac{2c}{t}$

normal: $y = t^2x + \frac{c}{t} - ct^3$

(5) Taylor expansions

(i) Maclaurin: $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$

(ii) Taylor I: $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$

(iii) Taylor II: $f(x+a) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \cdots$

[$x = 0$ gives the Maclaurin expansion]

(6) Hyperbolic Functions

$cosh^2x + sinh^2x = cosh2x$; $cosh^2x - sinh^2x = 1$

(7) Areas & Volumes

(i) Area of sector: $\frac{1}{2}r^2\theta$

(consider limit of area of triangle $\frac{1}{2}r^2 sin\theta$ as $\theta \to 0$)

(ii) Volume of sphere: $\frac{4}{3}\pi r^3$

(iii) Volume of pyramid or cone: $\frac{1}{3} \times$ base area $\times$ height

(8) Derivatives

$\frac{d}{dx}(a^x) = lna.a^x$

(9) Simpson's rule

$\int_a^b y\,dx \approx$

$$\frac{h}{3}\{(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\}$$

where $h = \frac{b-a}{n}$ (n even)

(10)

$\sqrt{2} = 1.4142135623730950488016887 \ldots$

$e = 2.7182818284590452353602874 \ldots$

Golden ratio: $\frac{1+\sqrt{5}}{2} = 1.618$ (4sf)

(11) Definitions

Whole numbers: $0, 1, 2, \ldots$

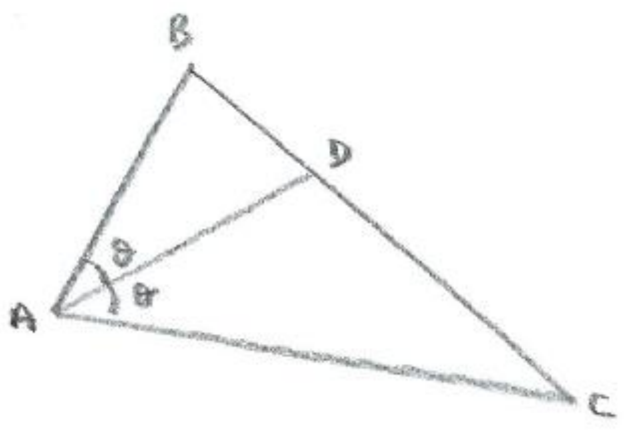
Natural numbers: no universal agreement;

usually $1, 2, 3, \ldots$ (ie $\mathbb{Z}^+$) but sometimes $0, 1, 2, \ldots$

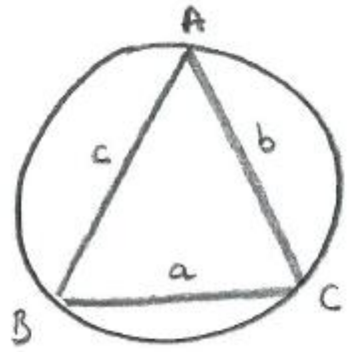
(12) Angle Bisector Theorem

Referring to the diagram below, the Angle Bisector theorem says that

$\frac{BD}{DC} = \frac{AB}{AC}$



(13) ABC is a triangle circumscribed by a circle of radius R, as shown in the diagram below.



As an extension of the Sine rule, $\frac{a}{sinA} = 2R$

(14)

(i) $(a + b + c)^3 = (a^3 + b^3 + c^3)$

$+3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b)$

$+6abc$

(ii) $(a + b + c)^4 = (a^4 + b^4 + c^4)$

$+4(a^3b + a^3c + b^3a + b^3c + c^3a + c^3b)$

$+6(a^2b^2 + a^2c^2 + b^2c^2) + 12(a^2bc + b^2ac + c^2ab)$

(iii) $(a + b + c)^n = \sum_{\substack{i,j,k \\ (i+j+k=n)}} \binom{n}{i,j,k} a^i b^j c^k$,

where $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$

(15) Reflection in the line $x = a$: $f(x) \to f(2a - x)$

(16) For the cubic $f(x) = ax^3 + bx^2 + cx + d$:

(i) There is always one point of inflexion, at $x = -\frac{b}{3a}$

(ii) Cubic curves have rotational symmetry (of order 2) about the PoI.

(iii) The (x-coordinate of the) PoI lies midway between any turning points.

(iv) The (x-coordinate of the) PoI is the average of the roots, when there are 3 real roots (and also when there are complex roots).

(v) There will be two turning points when $b^2 > 3ac$

(17) Heron's formula for the area of a triangle with sides a, b & c:

$\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$