Trigonometry - Part 2 (15 pages; 4/9/16)

(8) Solution of simple equations - for a range of angles

Example: To solve $\sin\left(2\theta - \frac{\pi}{6}\right) = 0.5$ for $0 < \theta \leq 2\pi$,

let $\phi = 2\theta - \frac{\pi}{6}$, so that $-\frac{\pi}{6} < \phi \leq 4\pi - \frac{\pi}{6}$

Then $\phi = \frac{\pi}{6}$, $\pi - \frac{\pi}{6}$, and also $\frac{\pi}{6} + 2\pi$ & $\pi - \frac{\pi}{6} + 2\pi$

and values for θ are obtained from $\theta = \frac{1}{2}(\phi + \frac{\pi}{6})$

[When $cos\phi = 0.5$, $\phi = \frac{\pi}{3}$, $2\pi - \frac{\pi}{3}$, and multiples of 2π can be added or subtracted.]

(9) General solution of simple equations

The approach above (for a limited range of θ) can be applied, to obtain two ('base') solutions in one cycle (except for $sin\theta = 1\ or - 1$). Then the general solutions can be derived as follows:

(i) For eg $sin\theta = 0.5$, the base solutions are $\theta = \frac{\pi}{6}$, $\pi - \frac{\pi}{6}$

A more concise alternative to $\frac{\pi}{6} + 2k\pi\ \ or\ \ \frac{5\pi}{6} + 2k\pi$ is to note that the solutions lie alternately $\frac{\pi}{6}$ ahead of and behind the multiples of π, so that we can write $\theta = n\pi + (-1)^n\left(\frac{\pi}{6}\right)$

So the general solution for $y = sinx$ is $x = n\pi + (-1)^n arcsiny$

(ii) For eg $cos\theta = 0.5$, the base solutions are $\theta = \frac{\pi}{3}$, $2\pi - \frac{\pi}{3}$, or alternatively $\theta = \frac{\pi}{3}$ & $-\frac{\pi}{3}$, from which we obtain the general solution of $\theta = 2k\pi \pm \frac{\pi}{3}$

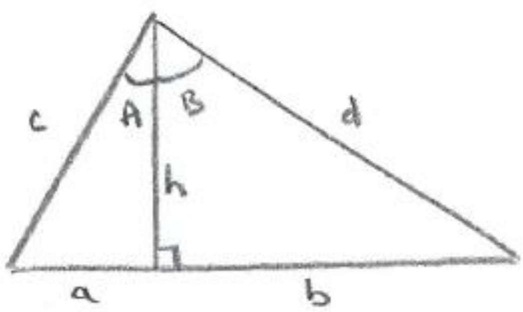
So the general solution for $y = cosx$ is $x = 2k\pi \pm arccosy$

(iii) For eg $tan\theta = 1$, the general solution is just $\theta = \frac{\pi}{4} + k\pi$

So the general solution for $y = tanx$ is $x = arctany + k\pi$

(10) Compound angle formulae

Proof of $sin(A + B) = sinAcosB + cosAsinB$:



Referring to the diagram,

the area of the large triangle can be formed in two ways:

$\frac{1}{2}cdsin(A + B) = \frac{1}{2}ah + \frac{1}{2}bh$

$\Rightarrow sin(A + B) = \frac{a}{c}.\frac{h}{d} + \frac{b}{d}.\frac{h}{c} = sinAcosB + sinBcosA$

The other compound angle formulae can be derived from the formula for $sin(A + B)$, as follows:

$sin(A - B) = sin(A + [-B]) = sinAcos(-B) + cosAsin(-B)$

$= sinAcosB - cosAsinB$

$cos(A + B) = sin(90 - [A + B]) = sin([90 - A] - B)$

$= sin(90 - A)cosB - cos(90 - A)sinB$

$= cosAcosB - sinAsinB$

$cos(A - B) = cos(A + [-B]) = cosAcos(-B) - sinAsin(-B)$

$= cosAcosB + sinAsinB$

$tan(A + B) = \frac{sin(A+B)}{cos(A+B)} = \frac{sinAcosB+cosAsinB}{cosAcosB-sinAsinB}$

$= \frac{tanA+tanB}{1-tanAtanB}$ (after dividing top and bottom by $cosAcosB$)

and similarly for $tan(A - B) = \frac{tanA-tanB}{1+tanAtanB}$

Proof using matrices

Rotation of θ followed by rotation of ϕ

$$\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} cos\phi & -sin\phi \\ sin\phi & cos\phi \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x'' \\ y'' \end{pmatrix}$$

$$\begin{pmatrix} cos\phi & -sin\phi \\ sin\phi & cos\phi \end{pmatrix}\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x'' \\ y'' \end{pmatrix}$$

Also $\begin{pmatrix} \cos(\theta + \phi) & -\sin(\theta + \phi) \\ \sin(\theta + \phi) & \cos(\theta + \phi) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x'' \\ y'' \end{pmatrix}$

Hence $\cos(\theta + \phi) = cos\theta cos\phi - sin\theta sin\phi$

& $\sin(\theta+\phi) = cos\theta sin\phi + sin\theta cos\phi$ or $sin\theta cos\phi + cos\theta sin\phi$

(11) $cos^2\theta$ *and* $sin^2\theta$

Using the results $cos^2\theta + sin^2\theta = 1$

(by applying Pythagoras to a right-angled triangle with sides $cos\theta, sin\theta$ & 1)

and $cos^2\theta - sin^2\theta = \cos(2\theta)$ (from the Compound angle formulae) we can derive:

$cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$ & $sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$

(12) $Rsin(\theta+\alpha)$ and $Rcos(\theta-\alpha)$

The compound angle formulae can be used to write the expression $asin\theta + bcos\theta$ in either of the forms $Rsin(\theta+\alpha)$ or $Rcos(\theta-\alpha)$.

Example

$\sqrt{3}sin\theta + cos\theta = Rsin(\theta+\alpha) = Rsin\theta cos\alpha + Rcos\theta sin\alpha$

and equating coefficients of $sin\theta$ & $cos\theta$ (given that the two expressions are to be equal for all values of θ),

$\sqrt{3} = Rcos\alpha$ & $1 = Rsin\alpha$,

so that $3+1 = R^2(cos^2\alpha + sin^2\alpha) = R^2$,

giving $R = 2$ (for example); and $tan\alpha = \frac{1}{\sqrt{3}}$, giving $\alpha = \frac{\pi}{6}$

Thus $\sqrt{3}sin\theta + cos\theta = 2sin\left(\theta + \frac{\pi}{6}\right)$

Alternatively,

$\sqrt{3}sin\theta + cos\theta\ = Rcos(\theta - \alpha) = Rcos\theta cos\alpha + Rsin\theta sin\alpha$,

so that $\sqrt{3} = Rsin\alpha$ & $1 = Rcos\alpha$, giving $R = 2$ again,

and $tan\alpha = \sqrt{3}$, so that $\alpha = \frac{\pi}{3}$

Thus $\sqrt{3}sin\theta + cos\theta$ can be written as either

$2sin\left(\theta + \frac{\pi}{6}\right)$ or $2\cos\left(\theta - \frac{\pi}{3}\right)$ [see note (b) below]

Had we chosen $Rsin(\theta - \alpha)$ instead, then

$\sqrt{3}sin\theta + cos\theta\ = Rsin\theta cos\alpha - Rcos\theta sin\alpha$,

giving $\sqrt{3} = Rcos\alpha$ & $1 = -Rsin\alpha$,

so that $R = 2$ and $tan\alpha = -\frac{1}{\sqrt{3}}$, or $\tan(-\alpha) = \frac{1}{\sqrt{3}}$,

with the same result as before.

Similarly,

$\sqrt{3}sin\theta + cos\theta = Rcos(\theta + \alpha) = Rcos\theta cos\alpha - Rsin\theta sin\alpha$,

giving $\sqrt{3} = -Rsin\alpha$ & $1 = Rcos\alpha$,

so that $R = 2$ and $tan\alpha = -\sqrt{3}$, or $\tan(-\alpha) = \sqrt{3}$, as before

Notes

(a) $Rsin(\theta - \alpha)$ can be chosen, if b is negative; similarly for $Rcos(\theta + \alpha)$ if a is negative.

(b) As $sin(\theta + \alpha) = \cos(90° - [\theta + \alpha]) = cos([90° - \alpha] - \theta)$

$= cos(\theta - [90° - \alpha])$, one form can be obtained from the other.

Similarly, $cos(\theta - \alpha) = \cos(\alpha - \theta) = sin(90° - [\alpha - \theta])$

$= sin(\theta + [90° - \alpha])$

(13) Factor formulae

$sin\theta + sin\phi = 2sinXcosY$

$sin\theta - sin\phi = 2cosXsinY$

$cos\theta + cos\phi = 2cosXcosY$

$cos\theta - cos\phi = -2sinXsinY$

where $X = \frac{1}{2}(\theta + \phi)$ & $Y = \frac{1}{2}(\theta - \phi)$

Proofs

Let $\theta = X + Y$ & $\phi = X - Y$

Then $sin\theta + sin\phi = \text{sinXcosY} + \text{cosXsinY} + \text{sinXcosY} - \text{cosXsinY}$

$= 2\text{sinXcosY}$, with X & Y as above; and similarly for the other formulae.

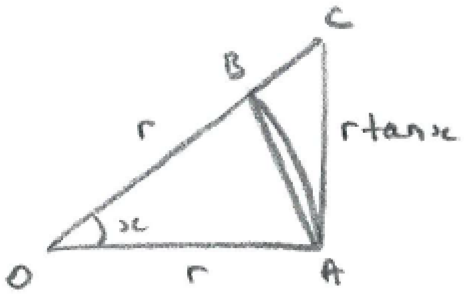
(14) Small angle approximations

These can be derived for $sinx$ & $cosx$ from their Maclaurin expansions [See " Maclaurin Series"].

Thus $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ ...\ \Rightarrow sinx \approx x$ for small x,

& $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \Rightarrow \cos x \approx 1 - \frac{x^2}{2}$ for small x

Alternative derivation



For $sinx \approx x$:

$\triangle OAB < sectorOAB < \triangle OAC$,

so that $\frac{1}{2}r^2 sinx < \frac{1}{2}r^2 x < \frac{1}{2}r(rtanx)$

and hence $1 < \frac{x}{sinx} < \frac{1}{cosx}$

As $x \to 0, \frac{1}{cosx} \to 1$ (from above);

ie we can make $\frac{1}{cosx}$ as close to 1 as we please,

and then $\frac{x}{sinx} \approx 1$; ie $sinx \approx x$ for small x

For $\cos x \approx 1 - \frac{x^2}{2}$:

Starting with $cos(2\theta) = cos^2\theta - sin^2\theta = 1 - 2sin^2\theta$,

and writing $x = 2\theta$, we have $cosx = 1 - 2sin^2\left(\frac{x}{2}\right)$

Then, for small x, $cosx \approx 1 - 2\left(\frac{x}{2}\right)^2 = 1 - \frac{x^2}{2}$

(15) Why are radians preferred to degrees?

(i) The key point is that $sin\theta \approx \theta$ for small θ measured in radians, but, if ϕ is measured in degrees, then

$$sin\phi = sin\theta \approx \theta = \left(\frac{\pi}{180}\right)\phi$$

So $\lim_{\theta \to 0} \frac{sin\theta}{\theta} = 1$ when θ is measured in radians

and $\lim_{\phi \to 0} \frac{sin\phi}{\phi} = \frac{\pi}{180}$ when ϕ is measured in degrees

(ii) Consider the graph of $sin\phi$, where ϕ is measured in degrees, and compare it with the graph of $sin\theta$, where θ is measured in radians. $sin\phi$ increases from 0 to 1 as ϕ increases from 0° to 90°, whereas $sin\theta$ increases from 0 to 1 as θ increases from 0 to $\frac{\pi}{2}$.

Thus the graph of $sin\phi$ is more stretched out than that of $sin\theta$, with a much smaller gradient (except when $cos\theta = cos\phi = 0$).

In particular, at the Origin, $y = sin\theta$ tends to $y = \theta$ only when θ is measured in radians.

(iii) When measuring angles in radians,

$$\frac{d}{d\theta}sin\theta = \lim_{h \to 0} \frac{\sin(\theta + h) - sin\theta}{h} = \lim_{h \to 0} \frac{sin\theta\cos(h) + cos\theta\sin(h) - sin\theta}{h}$$

As $\cos(h) \to 1$ as $h \to 0$, and $\frac{\sin(h)}{h} \to 1$, as we are measuring our angles in radians, $\frac{d}{d\theta}sin\theta = cos\theta$

But when measuring angles in degrees,

$$\frac{d}{d\phi}sin\phi = \lim_{h\to 0}\frac{sin\phi\cos(h)+cos\phi\sin(h)-sin\phi}{h} \text{ again,}$$

and it is still true that $\cos(h) \to 1$ as $h \to 0$, but now $\frac{\sin(h)}{h} \to \frac{\pi}{180}$

so that $\frac{d}{d\phi}sin\phi = \frac{\pi}{180}cos\phi$ (where ϕ is measured in degrees)

Note that, strictly speaking, it is not enough for ϕ to have a value which happens to be the number of degrees: the cosine (or sine) function itself is different, depending on whether the angle is measured in degrees or radians. To be clear, we could use the notation $sin_{deg}\phi$ and $sin_{rad}\theta$ (as in the next part).

(iv) Alternative derivation:

If ϕ is the angle in degrees, and θ is the angle in radians, so that $\phi = \left(\frac{180}{\pi}\right)\theta$, then

$$\frac{d}{d\phi}sin_{deg}\phi = \frac{d}{d\phi}sin_{rad}\theta = \left[\frac{d}{d\theta}sin_{rad}\theta\right]\frac{d\theta}{d\phi} = (cos_{rad}\theta)\left(\frac{\pi}{180}\right)$$

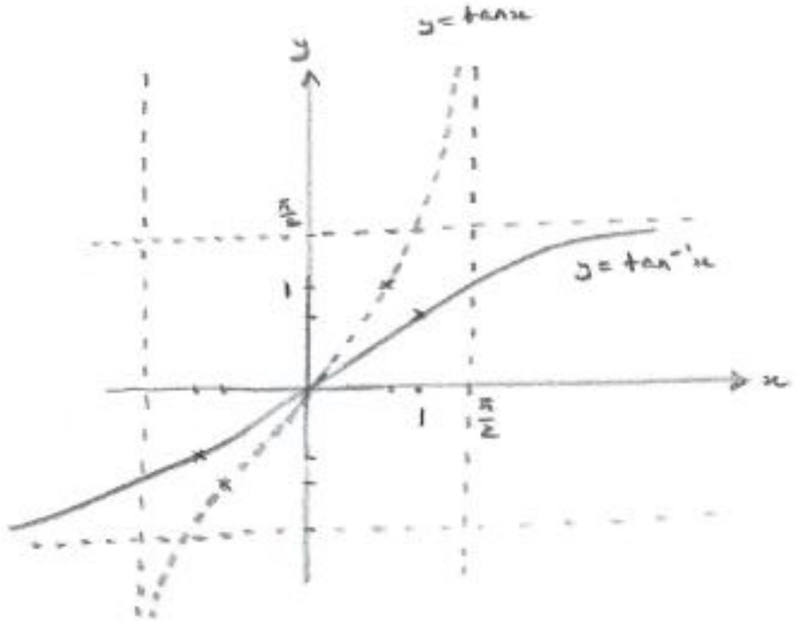
$$= (cos_{deg}\phi)\left(\frac{\pi}{180}\right)$$

Or: $\frac{d}{d\phi}sin_{deg}\phi = \frac{d}{d\phi}sin_{rad}\theta = \frac{d}{d\phi}sin_{rad}[\phi\left(\frac{\pi}{180}\right)]$

$$= \left(\frac{\pi}{180}\right)cos_{rad}\left[\phi\left(\frac{\pi}{180}\right)\right] = \left(\frac{\pi}{180}\right)cos_{rad}\theta \; = \left(\frac{\pi}{180}\right)cos_{deg}\phi$$

(16) Inverse Trig. Functions

(A) $y = tan^{-1}x$ $(or\ arctanx)$



(i) The scales have to be in radians in order for these graphs to be reflections of each other in $y = x$.

(ii) In order to establish the gradient of $y = tanx$ at the Origin:

$\frac{d}{dx}(\tan x) = sec^2x$ = 1 when $x = 0$

(this assumes that the angle is in radians)

[The above graph hasn't been drawn that well: the gradients of both $y = tanx$ and $y = tan^{-1}x$ are intended to be 1 at the Origin.]

(iii) In order for $y = tan^{-1}x$ to be a 1-1 mapping (and therefore a function), the domain of $y = tanx$ has to be limited to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

(iv) $\frac{d}{dx}\ tan^{-1}x$

Let $y = tan^{-1}x$

Then $tany = x$ and $sec^2y\ \frac{dy}{dx} = 1$ (differentiating implicitly wrt x) [alternatively, differentiate wrt y, to give $sec^2y\ = \frac{dx}{dy}$ and take the reciprocal]

And as $sec^2y\ = tan^2y + 1 = x^2 + 1$, $\frac{dy}{dx} = \frac{1}{1+x^2}$

Also $\int \frac{1}{1+x^2}\ dx = tan^{-1}x + c$

[See also "Integration methods".]

(v) Features of $\frac{dy}{dx}$ (in agreement with graph):

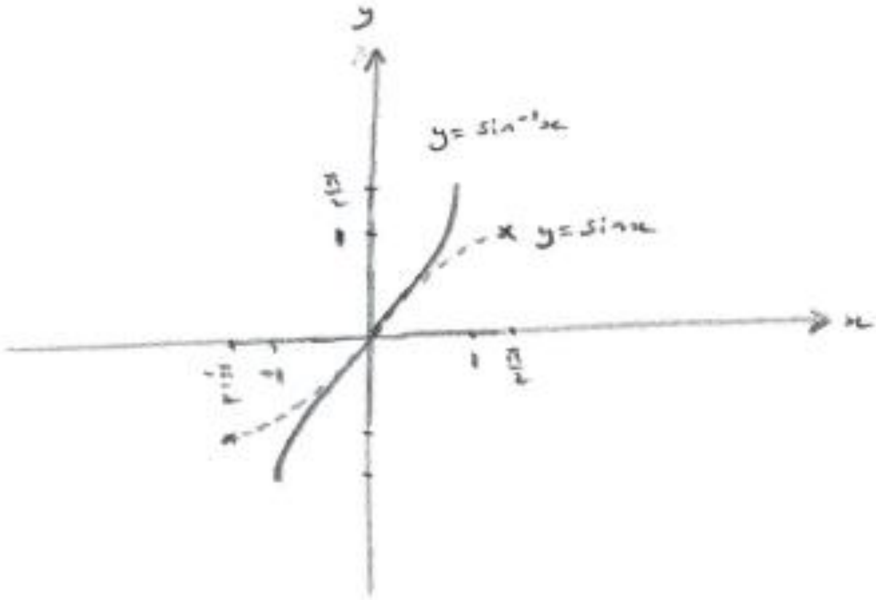
(a) $\frac{dy}{dx} \rightarrow 0$ as $x \rightarrow \pm\infty$

(b) $\frac{dy}{dx}$ is always positive

(c) $\frac{dy}{dx}$= 1 when $x = 0$

(vi) $tanx = y\ \rightarrow x = arctany + n\pi$

(B) $\boldsymbol{y = sin^{-1}x\ (or\ arcsinx)}$



(i) In order for $y = sin^{-1}x$ to be a 1-1 mapping (and therefore a function), the domain of $y = sinx$ has to be limited to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(ii) $\frac{d}{dx}(\sin x) = 1\ when\ x = 0$

(iii) $sinx = y\ \rightarrow x = \left(arcsiny\ or\ \pi - arcsiny\right) + 2n\pi$

or $x = n\pi + (-1)^n arcsiny$

(iv) $\frac{d}{dx}\ sin^{-1}x$

Let $y = sin^{-1}x$, so that $siny = x$ and $cosy\frac{dy}{dx} = 1$

Hence $\frac{dy}{dx} = \frac{1}{cosy} = \frac{1}{\sqrt{1-sin^2y}} = \frac{1}{\sqrt{1-x^2}}$, taking the positive root, as y is restricted to the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$, where $cosy > 0$ (assuming that $\frac{dy}{dx}$ is defined, so that $cosy \neq 0$)

(also $\frac{dy}{dx} > 0$ from the graph).

Also $\int \frac{1}{\sqrt{1-x^2}}\, dx = sin^{-1}x$ (see "Integration Methods").

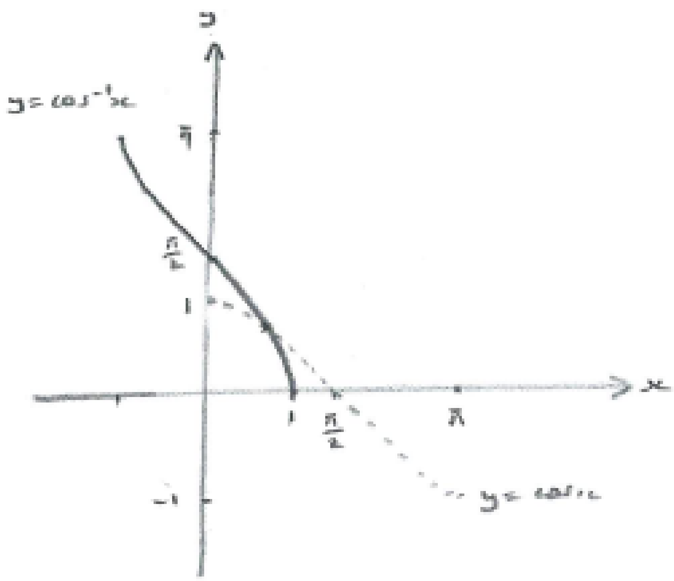
(v) Features of $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

(a) $\frac{dy}{dx} \to \infty$ as $x \to \pm 1$

(b) $\frac{dy}{dx}$ is always positive

(c) $\frac{dy}{dx}$ is undefined when $x \leq -1$ or $x \geq 1$

(C) $y = cos^{-1}x$ (or $arccosx$)



(i) In order for $y = cos^{-1}x$ to be a 1-1 mapping (and therefore a function), the domain of $y = cosx$ has to be limited to $[0\ ,\pi]$.

(ii) $cosx = y \ \rightarrow x = \left(arccosy \ \ or\ 2\pi - arccosy\right) + 2n\pi$

or $\pm arccosy + 2n\pi$

(iii) $\frac{d}{dx}\ cos^{-1}x$

Let $y = cos^{-1}x$, so that $cosy = x$ and $-siny\frac{dy}{dx} = 1$

Hence $\frac{dy}{dx} = \frac{-1}{siny} = \frac{-1}{\sqrt{1-cos^2y}} = \frac{-1}{\sqrt{1-x^2}}$

taking the positive root , as y is restricted to the range $[0\ ,\ \pi]$,

when $siny > 0$ (assuming that $\frac{dy}{dx}$ is defined, so that $siny \neq 0$)

(also $\frac{dy}{dx} < 0$ from the graph)

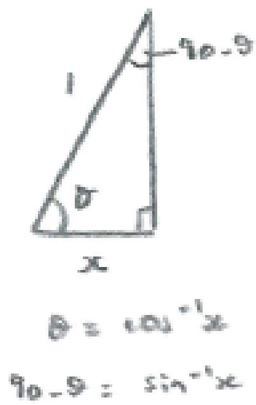
Also $\int \frac{1}{\sqrt{1-x^2}}\ dx = -cos^{-1}x$ (see "Integration Methods").

Note: $cos^{-1}x + sin^{-1}x = \frac{\pi}{2}$ (see diagram),

so that $-cos^{-1}x + c = sin^{-1}x - \frac{\pi}{2} + c$

Thus the two alternative expressions for $\int \frac{1}{\sqrt{1-x^2}}\ dx$

($sin^{-1}x$ & $-\ cos^{-1}x$) differ by a constant.



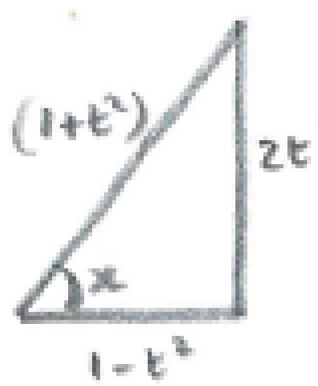
(iv) $y = cos^{-1}x$ is the reflection of $y = sin^{-1}x$ in $y = \frac{\pi}{4}$, $since$

$cos^{-1}x + sin^{-1}x = \frac{\pi}{2}$

(17) $t = tan(\frac{x}{2})$ substitution

$t = tan\left(\frac{x}{2}\right) \Rightarrow \tan x = \frac{2t}{1-t^2}$ (by double angle formula)

Other trig. ratios can be read off the right-angled triangle in the diagram:



This substitution can be useful (mainly as a method of last resort) for awkward integrals - enabling a complicated combination of trig. ratios to be converted to a combination of powers of t. Equations can also sometimes be solved by this method.

[See "Integration Methods" & "Trig. Exercises".]