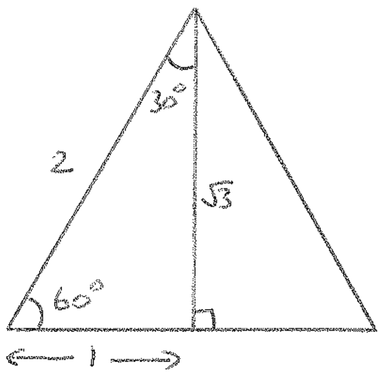
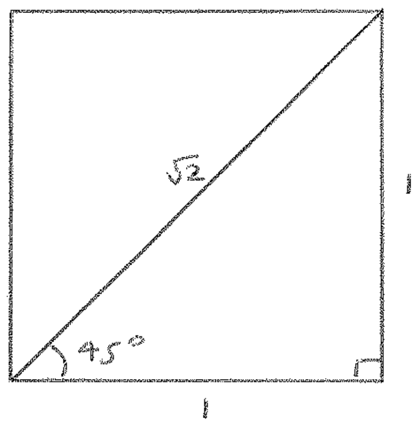
Trigonometry - Part 1 (12 pages; 4/9/16)

(1) Sin, cos & tan of $30°$, $60°$ & $45°$





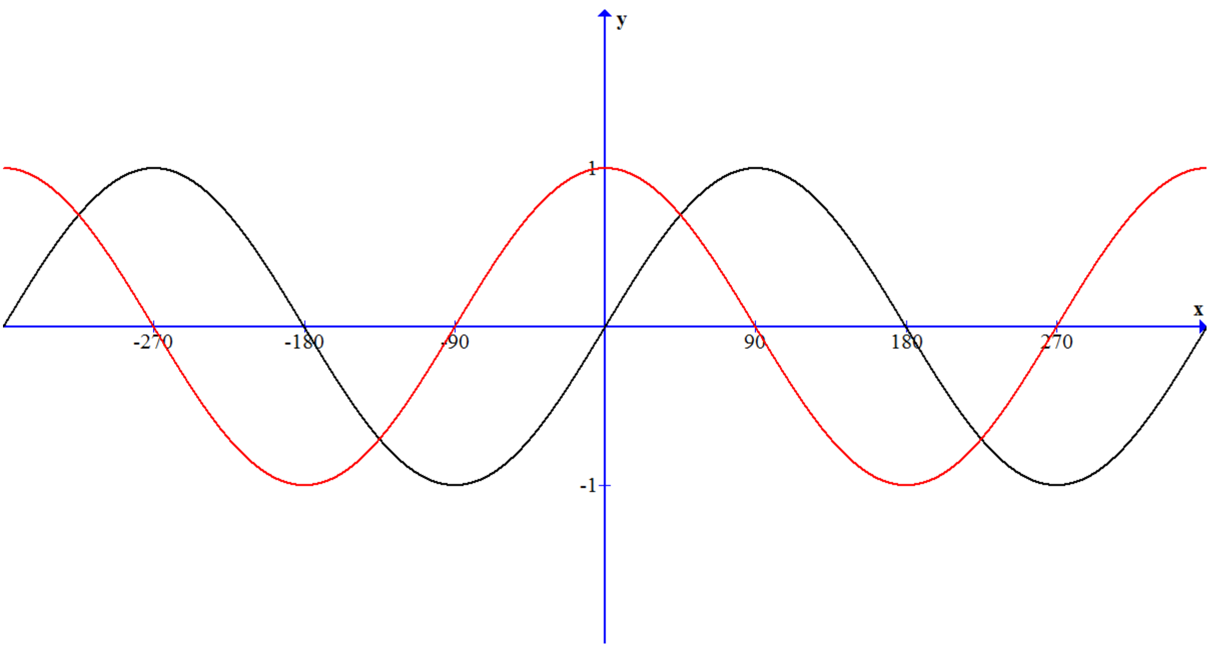
$sin30° = \frac{1}{2}$; $sin60° = \frac{\sqrt{3}}{2}$

$cos30° = \frac{\sqrt{3}}{2}$; $cos60° = \frac{1}{2}$

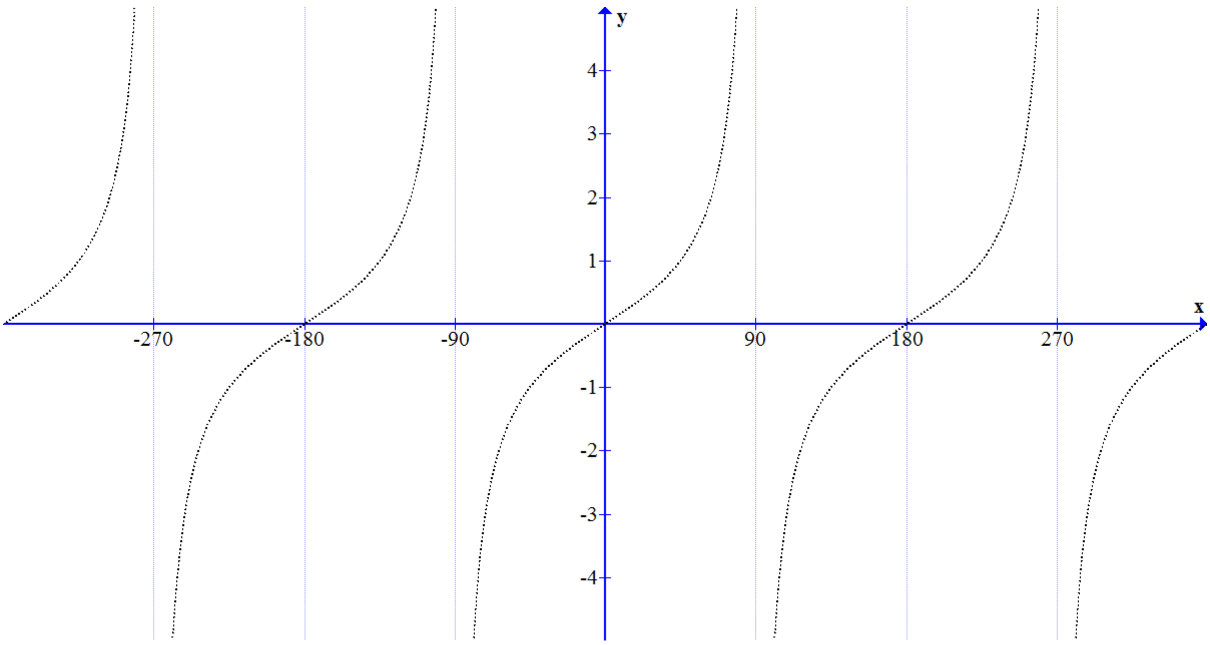
$tan30° = \frac{1}{\sqrt{3}}$; $tan60° = \sqrt{3}$

$cos45° = sin45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$tan45° = 1$



Graphs of $y = sinx$ $(black)$ & $y = cosx$ (red)



Graph of $y = tanx$

As $\frac{\sqrt{3}}{2}$ is larger than $\frac{1}{2}$, the shape of the sine curve makes it clear that $sin30° = \frac{1}{2}$ and $sin60° = \frac{\sqrt{3}}{2}$, rather than the other way round; and similarly for the cosine curve.

Also note that, since $tan45° = 1$ and the tangent function is increasing , we would expect $tan30°$ to be less than 1 and $tan60°$ to be greater than 1 (so that there should be no confusion as to which is $\frac{1}{\sqrt{3}}$ and which is $\sqrt{3}$).

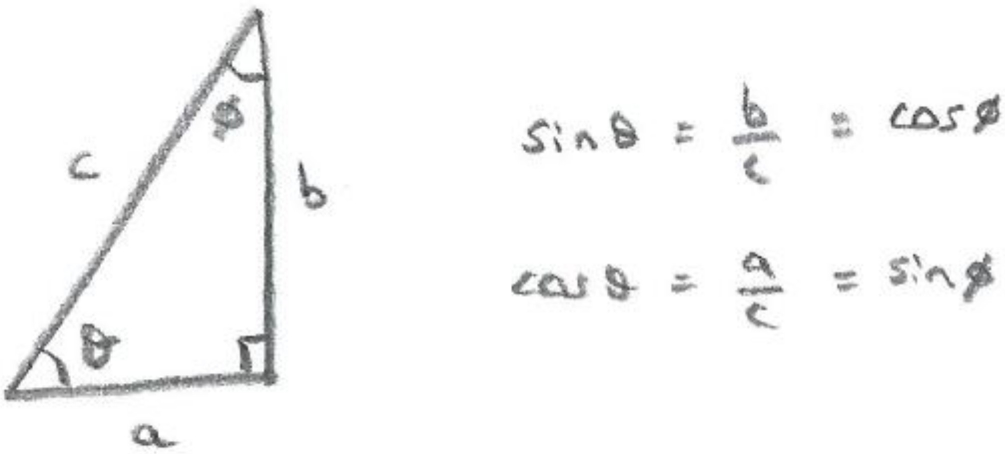
sin, cos & tan of multiples of $30°, 45°$ & $60°$ can be found by referring to the graph.

(2) Symmetry

(i) $\cos(-\theta) = cos\theta$ & $\sin(-\theta) = -sin\theta$ (from the graphs)

(ii) $\sin(180° - \theta) = sin\theta$ and $\cos(360° - \theta) = cos\theta$, by the symmetries of the sine and cosine graphs about $\theta = 90°$ and 180°, respectively.

(3) Complementary angles

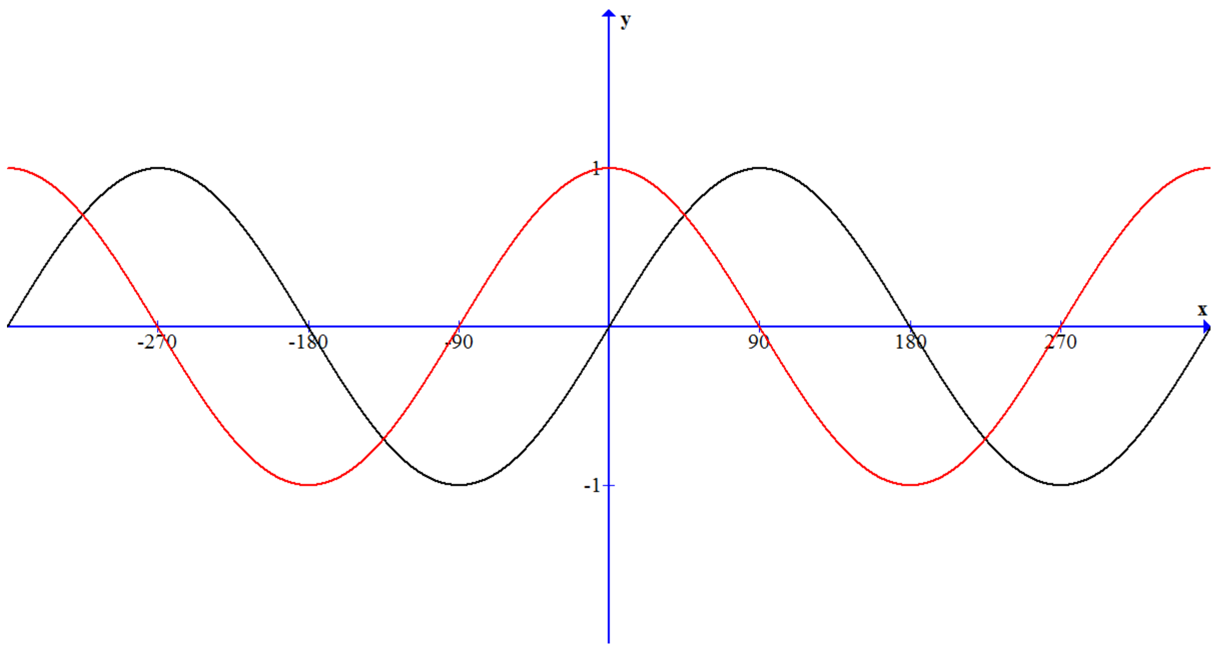


$sin\theta = cos\phi = \cos(90° - \theta)$ and $cos\theta = sin\phi = sin(90° - \theta)$

θ & ϕ are 'complementary' angles (ie they add up to 90°). This is the origin of the term 'cosine': $cos\theta$ is the sine of the angle complementary to θ.

Similarly, $cot\theta = tan\phi,$ and $cosec\theta = \frac{1}{sin\theta} = \frac{1}{cos\phi} = sec\phi$

(4) Translations



Graphs of $y = sinx\ (black)$ & $y = cosx\ (red)$

[Note that replacing θ with $\theta + \alpha$ produces a translation of α to the left, and replacing θ with $\theta - \alpha$ produces a translation of α to the right. See "Transformations of Functions" for further details.]

As an alternative to using the compound angle formulae (see Part 2), the following examples can be tackled by considering the translation and/or reflection involved; or often just by examining the graph.

Examples

(i) $\sin(360° - \theta) = \sin(-\theta) = -sin\theta$ (or from the graph)

(ii) $\sin(\theta + 180°) = \sin(\theta - 180°) = -\sin(180° - \theta) = -sin\theta$

(or from the graph, or by noting that replacing θ in $sin\theta$ by

$\theta + 180°$ produces a translation of 180° to the left, which gives the graph of $-sin\theta$)

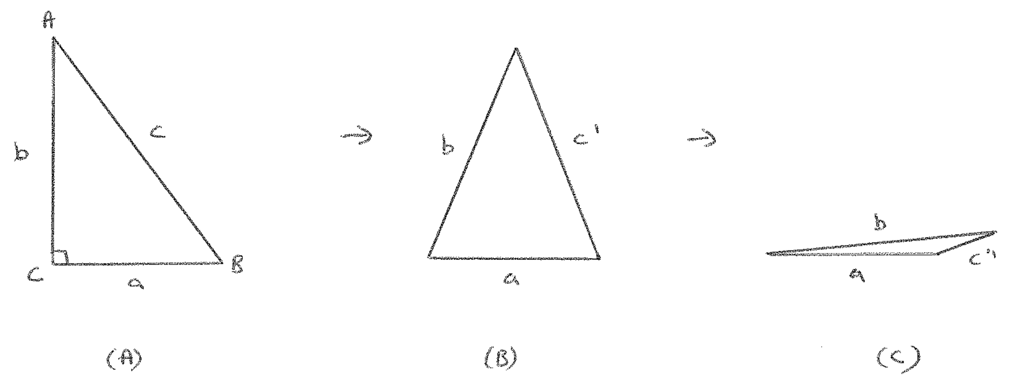
(iii) $\cos(180° - \theta) = \cos(\theta - 180°)$, which is obtained from $cos\theta$

by a translation of 180° to the right, which is seen to be $-cos\theta$

(as can be verified from the compound angle formula).

(iv) $sin(\theta + 90°)$ can be obtained from $sin\theta$ by a translation of 90° to the left, which is seen to be $cos\theta$

(5) Limiting cases of the Cosine rule



Start with a right-angled triangle with sides a, b & c (as in diagram A), such that c is the hypotenuse and side b is longer than side a.

This then gives $c^2 = a^2 + b^2$, by Pythagoras' Theorem.

If the angle C is reduced from being a right-angle, such that the lengths a & b remain the same, then the side c is also reduced (to c' in diagram B).

If C is reduced to zero, then c becomes $c'' = b - a$ (diagram C shows the position as C approaches zero).

If we now consider the adjustment of $-2abcosC$ to the formula for c^2 , we can see that it has the following necessary properties:

(i) it is zero if $C = 90°$

(ii) the adjustment becomes a larger negative value as C reduces

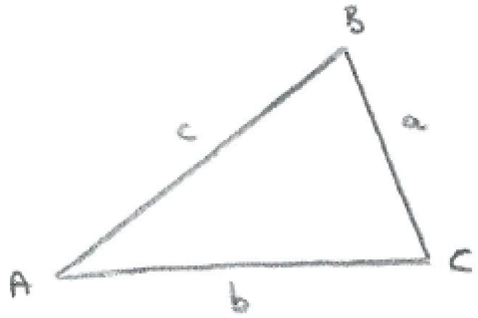
(iii) the adjustment is symmetrical in a & b

(iv) when $C = 0°$, the adjustment becomes $-2ab$, giving

$c^2 = a^2 + b^2 - 2ab = (b - a)^2$, so that $c = b - a$ (if $b > a$)

(6) Applying the Sine & Cosine rules

As $sin\theta = sin(180° - \theta)$, care should be taken when using the Sine rule to find an angle in a triangle that is close to 90°. The problem can sometimes be avoided by finding other angles first and subtracting from 180°. There is never any risk with using the Cosine rule, if that is feasible for the given problem.



For the triangle above, the following combinations of sides and angles will always enable the other sides and angles to be determined uniquely (ie any two triangles thus created will be congruent, with a reflection in the plane of the paper being allowed):

(I) a, b & c known

(II) A, B (and hence C) & a (eg) known

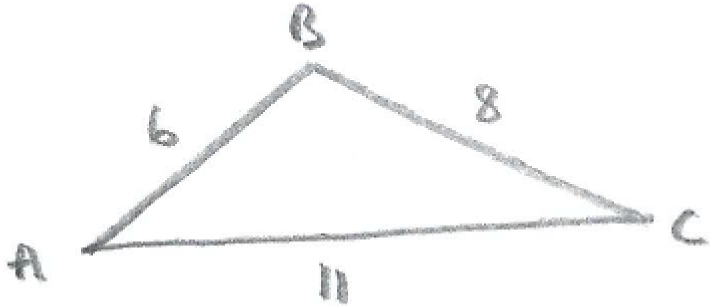
(III) a, b & C ($or\ b, c$ & $A\ etc$) known

For the following case, there will be two solutions if A is acute, $a < b$ and $B \neq 90°$:

(IV) a, b, A $(or\ B)$ known

For case (I), where a, b & c are known, the following strategy can be applied for finding the angles:

Apply the Cosine rule to find the angle opposite the largest side (or one of the largest sides, in the case of an isosceles triangle). [The reason for this will become clear from the example below.] Then apply either the Cosine or Sine rules to find another angle (the Cosine rule avoids having to take the sine of a rounded angle). Avoid using the Sine rule to find an angle that could be greater than 90° (giving two possible answers - though only one of them will be correct).



Referring to the above diagram, B (the largest angle) can be determined from the Cosine rule:

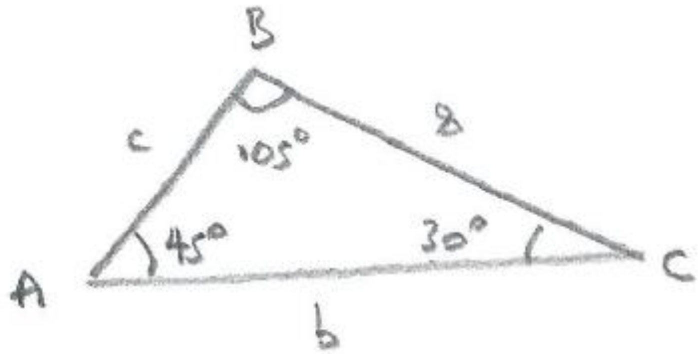
$11^2 = 6^2 + 8^2 - 2(6)(8)cosB$, giving $B = 102.636°$

Then A can be determined either by applying the Cosine rule again, or from the Sine rule:

$\frac{sinA}{8} = \frac{sin102.636°}{11}$

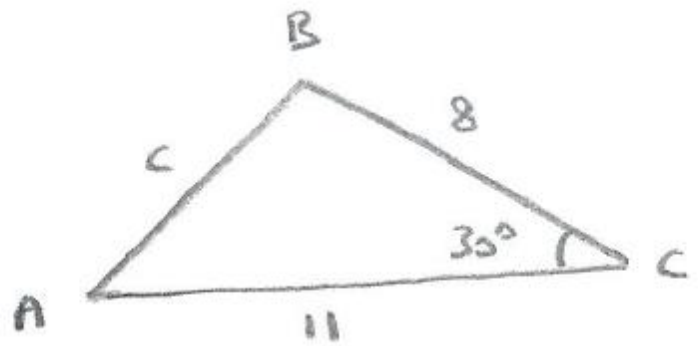
(Had we used the Cosine rule to obtain A (say), rather than B, then it would be unsafe to use the Sine rule to obtain B, since we can't be sure that B is greater than 90°. We could however find C from the Sine rule, and deduce B from A and C; or of course use the Cosine rule again to find either A or B.)

For case (II), where A, B (and hence C) & a are known, simply apply the Sine rule twice (as there is no ambiguity when finding a side from an angle).



Referring to the above diagram, $\frac{b}{sin105°} = \frac{8}{sin45°} = \frac{c}{sin30°}$

For case (III), where a, b & C are known, apply the Cosine rule to find c; then either the Cosine or Sine rules to find another angle (choosing the one opposite the smaller side if using the Sine rule).

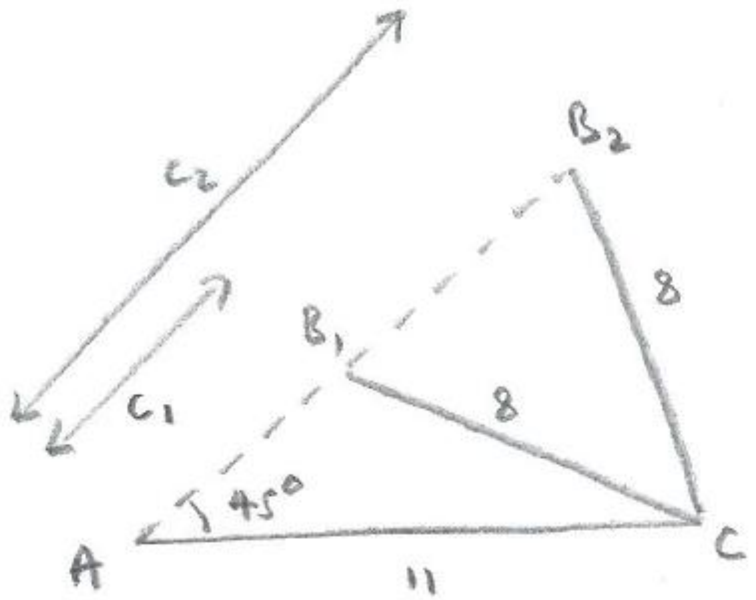


Referring to the above diagram, $c^2 = 11^2 + 8^2 - 2(11)(8)cos30°$,

giving $c = 5.70785$

Then $\frac{sinA}{8} = \frac{sin30°}{5.70785}$ to find A (but not $\frac{sinB}{11} = \frac{sin30°}{5.70785}$, as we don't know whether B is greater or less than 90°). Alternatively, the Cosine rule can be used again to find A or B.

For case (IV), where a, b & A are known, use the Sine rule to find B. If $a < b$, there will be two possible values for B (unless $B = 90°$). If $a \geq b$, B will be acute.



Referring to the above diagram, $\frac{sin45°}{8} = \frac{sinB}{11}$, giving $sinB = 0.97227$

and $B = 76.476°$ or $103.525°$

Alternatively, the Cosine rule could be used to find the two possible values for c:

$$8^2 = 11^2 + c^2 - 2(11)c(cos45°)$$

(7) Radians

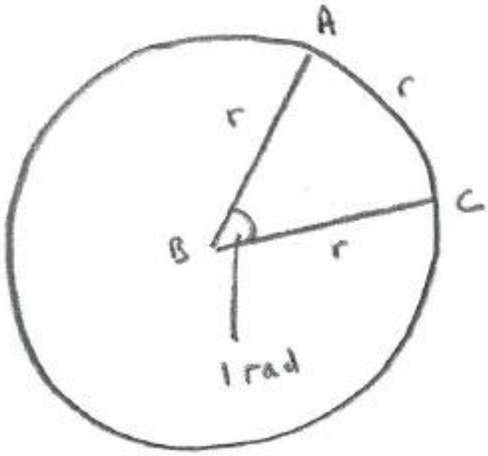


Fig. 1

In Fig. 1, the arc length AC equals the radius of the circle, and the angle ABC is defined to be 1 radian.

The chord AC is just smaller than r. Hence the triangle ABC is a slightly squashed equilateral triangle, and so 1 rad is just less than 60^0 .

The exact size can be determined by proportional reasoning, using the table below; together with other useful facts.

	angle (deg)	angle (rad)	arc length	area of sector
1	a	1	r	
2	360	b	$2\pi r$	πr^2
3	c	θ	d	e
4	ϕ	f		

Line 1 is based on the definition of the radian.

Line 2 is based on what we know about the circumference and area of a circle.

From the arc length column, we see that line 2 is 2π times line 1.

Thus $a = \frac{360}{2\pi} = \frac{180}{\pi} = 57.3° \ (3sf)$; ie 1 radian is approx. 57.3°

And $b = 2\pi(1) = 2\pi$; ie there are 2π radians in a circle.

Then, as line 3 is θ times line 1, $c = \theta a = \theta\left(\frac{360}{2\pi}\right)$ or $\theta\left(\frac{180}{\pi}\right)$; ie we can convert from radians to degrees by multiplying by $\frac{360}{2\pi}$ (or $\frac{180}{\pi}$).

Also, $d = \theta r$.

Then noting, from the arc length column, that line 3 is $\frac{\theta r}{2\pi r}$ times line 2,

$e = \left(\frac{\theta r}{2\pi r}\right)(\pi r^2) = \frac{1}{2}\theta r^2$, which is the area of a sector with an angle of θ rad.

[As an aid to memory, the triangle ABC in Fig. 1 has area $\frac{1}{2}r^2 sin\theta$, and as $sin\theta \rightarrow \theta \ as \ \theta \rightarrow 0$ [see "Small Angle Approximations" in Part 2], this area tends to $\frac{1}{2}r^2\theta$]

Finally, $f = (1)\left(\frac{\phi}{a}\right) = \left(\frac{2\pi}{360}\right)\phi$, to give the radian equivalent of an angle in degrees.

[As the angle in degrees is much larger than the corresponding angle in radians, there should never be any doubt whether to multiply or divide by $\frac{360}{2\pi}$ when switching between degrees and radians.]

[See Part 2 for "Why are radians preferred to degrees?"]