Trigonometry - Exercises (Sol'ns) (12 pages; 7/10/18)

(1) Solve the equation $sinx - cosx = 0.5,\ for\ 0° < x < 360°$

Solution

Method 1

Write $sinx - cosx = Rsin(x - \alpha) = R(sinxcos\alpha - cosxsin\alpha)$,

so that $Rcos\alpha = 1$ & $Rsin\alpha = 1$,

and hence $R^2(cos^2\alpha + sin^2\alpha) = 2$, so that $R = \sqrt{2}$

Also $tan\alpha = 1$, so that $\alpha = 45°$ (for example).

Thus the original equation becomes $\sqrt{2}sin(x - 45°) = 0.5$

Then let $u = x - 45°$, so that $-45° < u < 315°$

$sinu = \frac{1}{2\sqrt{2}} \Rightarrow u = 20.70481\ or\ 180 - 20.70481$

(and there are no other solutions within the range for u)

So $x = u + 45° = 65.7°\ or\ 204.3°$ (1dp)

Method 2

$sinx - cosx = 0.5 \Rightarrow tanx - 1 = 0.5secx$

$\Rightarrow (tanx - 1)^2 = \frac{sec^2x}{4}$, if we exclude solutions of $tanx - 1 = -0.5secx$

$\Rightarrow 4(tan^2x - 2tanx + 1) = 1 + tan^2x$

$\Rightarrow 3tan^2x - 8tanx + 3 = 0$

$\Rightarrow tanx = \frac{8\pm\sqrt{28}}{6} = \frac{1}{3}(4 \pm \sqrt{7}) = 2.21525\ or\ 0.45142$

$\Rightarrow x = 65.7° \; or \; 24.3°$,

as well as $65.7 + 180 = 245.7°$ and $24.3 + 180 = 204.3°$

But $24.3° \; and \; 245.7°$ are solutions of $tanx - 1 = -0.5secx$ and can therefore be excluded.

Thus the solutions are $x = 65.7° \; or \; 204.3°$

Method 3

$sinx - cosx = 0.5 \Rightarrow sin^2x = (cosx + 0.5)^2$

but this will include solutions of $-sinx - cosx = 0.5$, which will need to be removed

$\Rightarrow 1 - cos^2x = cos^2x + cosx + \frac{1}{4}$

$\Rightarrow 2cos^2x + cosx - \frac{3}{4} = 0$

$\Rightarrow 8cos^2x + 4cosx - 3 = 0$

$\Rightarrow cosx = \frac{-4 \pm \sqrt{16+96}}{16} = \frac{-1 \pm \sqrt{7}}{4} = -0.91144$ or 0.41144

$\Rightarrow x = 155.7°, 360 - 155.7 = 204.3°, 65.7°$

$or \; 360 - 65.7 = 294.3°$

The only solutions of the required equation are

$x = 65.7° \; and \; 204.3°$

(the other two are found to be solutions of $-sinx - cosx = 0.5$)

Method 4

$t = tan\left(\frac{x}{2}\right) \Rightarrow cosx = \frac{1-t^2}{1+t^2}$ & $sinx = \frac{2t}{1+t^2}$ (standard results - see "Trigonometry - Part 2")

Then, substituting into our equation:

$\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} = \frac{1}{2}$

$\Rightarrow 2\{2t - (1 - t^2)\} = 1 + t^2 \quad \Rightarrow t^2 + 4t - 3 = 0$

$\Rightarrow t = \frac{-4 \pm \sqrt{28}}{2} = -2 \pm \sqrt{7} = 0.64575 \; or - 4.64575$

$\Rightarrow \frac{x}{2} = 32.852° \; or - 77.852° + 180°$

(these are the only values between $0° \; and \; 180°$, which is the permissible range for $\frac{x}{2}$)

and hence $x = 65.7° \; or \; 204.3°$ (1dp)

(2) If the point (x, y) is rotated (anti-clockwise) about the Origin by an infinitesimal angle $\delta\theta$ (radians), show that the changes in the coordinates are given by: $\delta x = -y\delta\theta$ & $\delta y = x\delta\theta$

Solution

$\delta x = rcos(\theta + \delta\theta) - rcos\theta$

$= rcos\theta cos\delta\theta - rsin\theta sin\delta\theta - rcos\theta$

$= rcos\theta\left(1 - \frac{1}{2}(\delta\theta)^2 + \cdots\right) - rsin\theta\left(\delta\theta - \frac{1}{6}(\delta\theta)^3 + \cdots\right) - rcos\theta$

$= -rsin\theta(\delta\theta)$ to 1st order in $\delta\theta$

$= -y\delta\theta$, as required

And $\delta y = rsin(\theta + \delta\theta) - rsin\theta$

$= rsin\theta cos\delta\theta + rcos\theta sin\delta\theta - rsin\theta$

$= rsin\theta\left(1 - \frac{1}{2}(\delta\theta)^2 + \cdots\right) + rcos\theta\left(\delta\theta - \frac{1}{6}(\delta\theta)^3 + \cdots\right) - rsin\theta$

$= rcos\theta(\delta\theta)$ to 1st order in $\delta\theta$

$= x\delta\theta$, as required

(3) Given that $cos^5\theta = \frac{1}{16}(cos5\theta + 5cos3\theta + 10cos\theta)$ and

$cos^6\theta = \frac{1}{32}(cos6\theta + 6cos4\theta + 15cos2\theta + 10)$,

find expressions for $sin^5\theta$ and $sin^6\theta$

Solution

$sin^5\theta = cos^5\left(\frac{\pi}{2} - \theta\right)$

$= \frac{1}{16}(\cos[5\left(\frac{\pi}{2} - \theta\right)] + 5\cos[3(\frac{\pi}{2} - \theta)] + 10\cos(\frac{\pi}{2} - \theta))$

$= \frac{1}{16}(\cos[\frac{\pi}{2} - 5\theta] + 5\cos[-\frac{\pi}{2} - 3\theta] + 10\sin\theta)$

$= \frac{1}{16}(\sin 5\theta + 5\cos(\frac{\pi}{2} + 3\theta) + 10\sin\theta)$

$= \frac{1}{16}(\sin5\theta + 5\cos(\frac{\pi}{2} - [-3\theta]) + 10\sin\theta)$

$= \frac{1}{16}(sin5\theta + 5\sin(-3\theta) + 10sin\theta)$

$= \frac{1}{16}(sin5\theta - 5\sin3\theta + 10sin\theta)$

And $sin^6\theta = cos^6\left(\frac{\pi}{2} - \theta\right)$

$= \frac{1}{32}(\cos[6\left(\frac{\pi}{2} - \theta\right)] + 6\cos[4\left(\frac{\pi}{2} - \theta\right)] + 15\cos[2\left(\frac{\pi}{2} - \theta\right)] + 10)$

$= \frac{1}{32}(\cos(\pi - 6\theta) + 6\cos(-4\theta) + 15\cos(\pi - 2\theta) + 10)$

$= \frac{1}{32}(-cos6\theta + 6cos4\theta - 15cos2\theta + 10)$

(4) Express $-cos\theta$ in the form $cos\alpha$ (where α is to be found in terms of θ), using an algebraic method.

Solution

$-cos\theta = -\sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\theta - \frac{\pi}{2}\right)$

$= \cos(\frac{\pi}{2} - \left[\theta - \frac{\pi}{2}\right]) = \cos(\pi - \theta)$ (or $\cos(3\pi - \theta)$ etc)

Alternatively, $-cos\theta = -\cos(-\theta) = -\sin\left(\frac{\pi}{2} - [-\theta]\right)$

$= \sin\left(-\frac{\pi}{2} - \theta\right) = \cos\left(\frac{\pi}{2} - \left[-\frac{\pi}{2} - \theta\right]\right) = \cos(\pi + \theta)$

(or $\cos(3\pi + \theta)$ etc)

(5) Simplify $\sqrt{2(1 - cos\theta)}$ and $\sqrt{2(1 + cos\theta)}$

Solution

$\cos\theta = cos^2(\theta/2) - sin^2(\theta/2) = 1 - 2sin^2(\theta/2)$

so that $1 - \cos\theta = 2sin^2(\theta/2)$ and $\sqrt{2(1 - cos\theta)} = 2\sin(\theta/2)$

Also, $\cos2\theta = cos^2\theta - sin^2\theta = 2cos^2\theta - 1$, so that

$1 + \cos\theta = 2cos^2\left(\frac{\theta}{2}\right)$ and $\sqrt{2(1 + cos\theta)} = 2\cos(\frac{\theta}{2})$

(6) Show that

(i) $cos^4\theta - sin^4\theta = cos2\theta$

(ii) $cos^4\theta + sin^4\theta = 1 - \frac{1}{2}sin^2(2\theta)$

Solution

(i) $cos^4\theta - sin^4\theta = (cos^2\theta - sin^2\theta)(cos^2\theta + sin^2\theta)$

$= cos2\theta(1) = cos2\theta$

(ii) Consider

$1 = (cos^2\theta + sin^2\theta)^2 = cos^4\theta + sin^4\theta + 2cos^2\theta sin^2\theta$

Then $cos^4\theta + sin^4\theta = 1 - 2cos^2\theta sin^2\theta = 1 - \frac{1}{2}(2cos\theta sin\theta)^2$

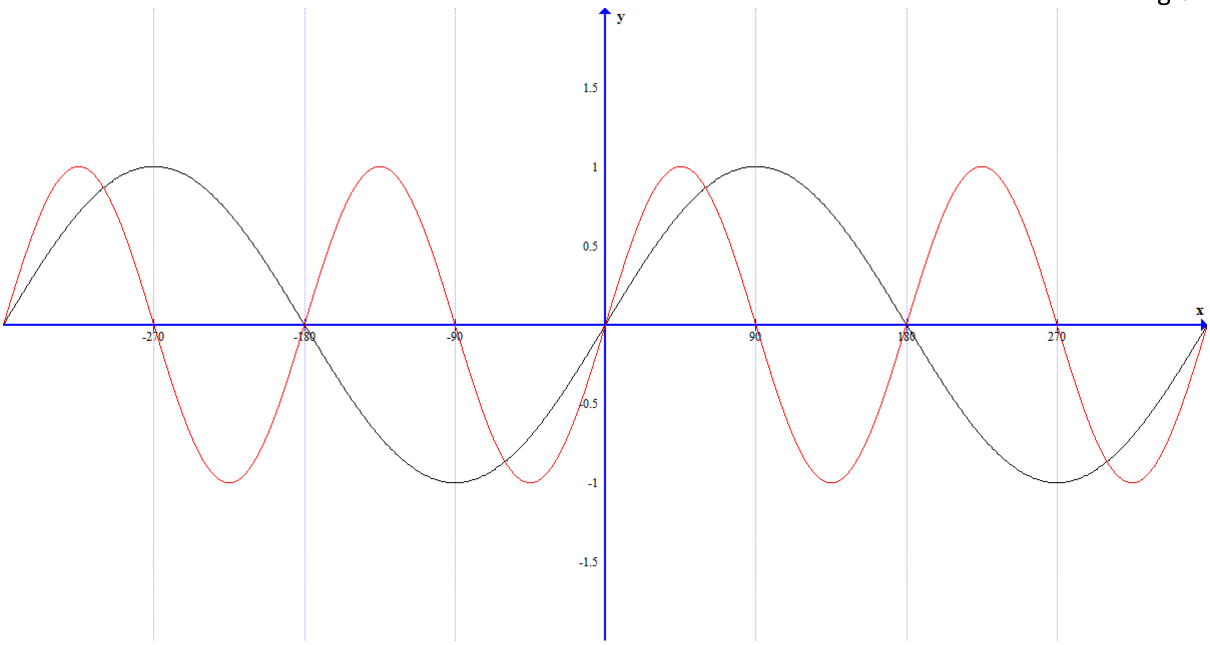
$= 1 - \frac{1}{2}sin^2(2\theta)$, as required.

(7) Sketch $y = \sin(2x + 30°)$

Solution

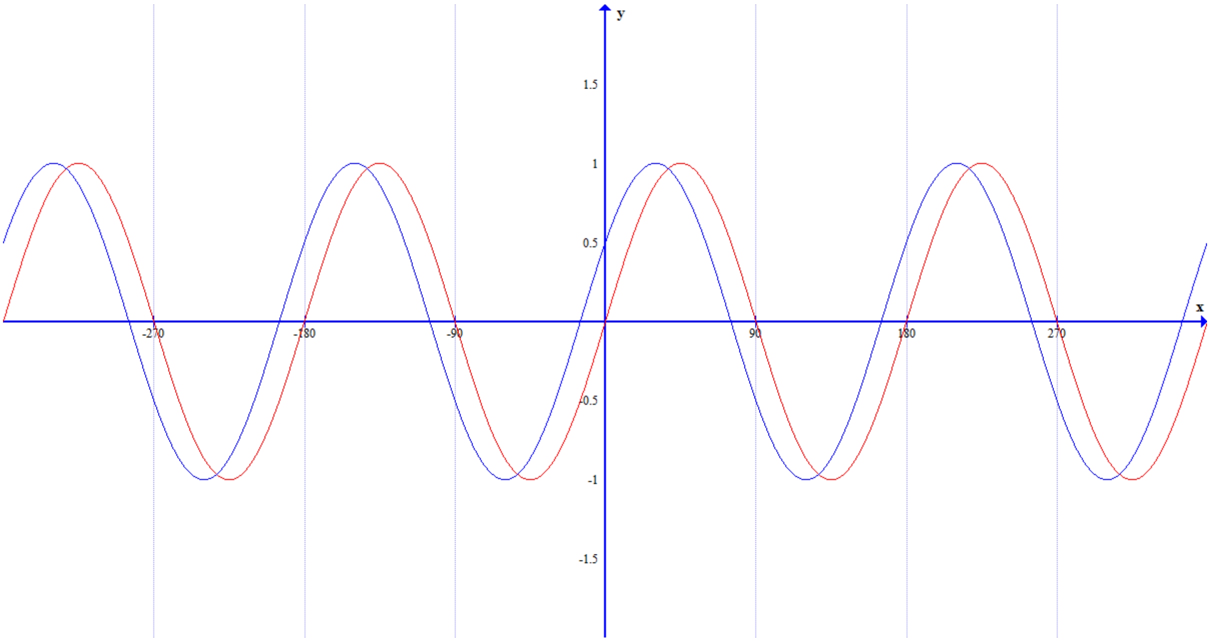
This is a composite transformation of $y = sinx$, and we have a choice of two approaches:

(i) $y = sinx \rightarrow y = sin2x$ [stretch of factor $\frac{1}{2}$ in the x-direction]

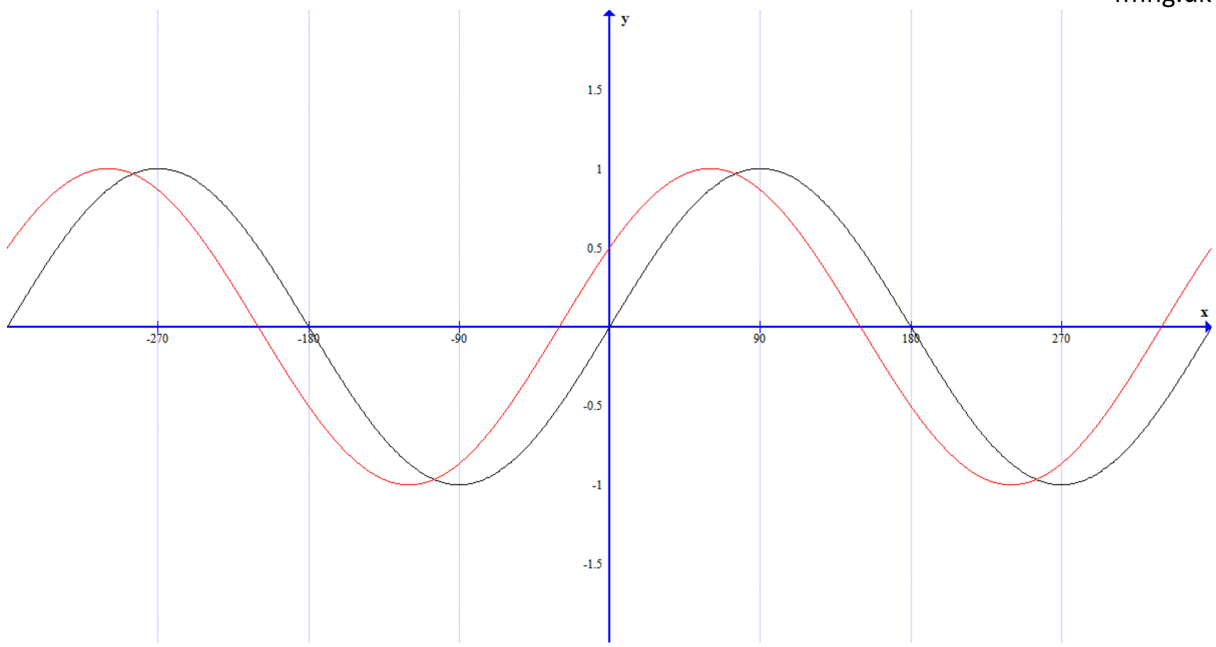


$\rightarrow y = \sin(2[x + 15°]$ [translation of 15° to the left]

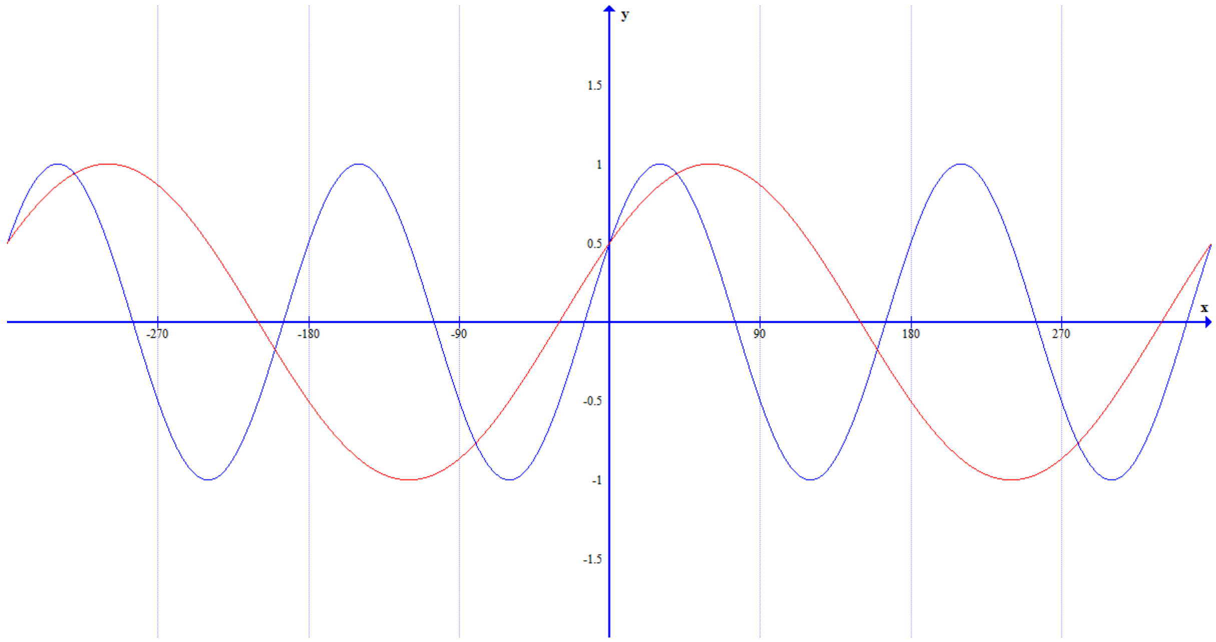
$= \sin(2x + 30°)$



(ii) $y = sinx \rightarrow y = \sin(x + 30°)$ [translation of 30° to the left]



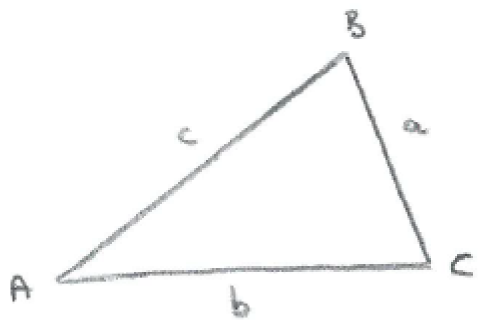
$\rightarrow y = \sin(2x + 30)$ [stretch of factor $\frac{1}{2}$ in the x-direction]



Note that, in the above transformation, the graph 'pivots' about $x = 0$; ie $\sin(2x + 30°) = \sin(x + 30°)$ at $x = 0$.

You may find approach (i) easier to carry out.

(8) Denote the sides of a triangle by a, b & c, and the angles (opposite these sides respectively) by A, B & C, as in the diagram below.



(i) What combinations of sides and angles will always enable the other sides and angles to be determined uniquely (ie any two triangles thus created will be congruent, so that a reflection in the plane of the paper is allowed)? What combination gives rise to two possibilities in some cases?

(ii) When finding missing lengths and angles, what ambiguous situation can arise (apart from the 2 solutions in IV), and how can it be avoided?

Solution

(i)

(I) a, b & c known ⇒ unique solution

(II) A, B (and hence C) & a (eg) ⇒ unique solution

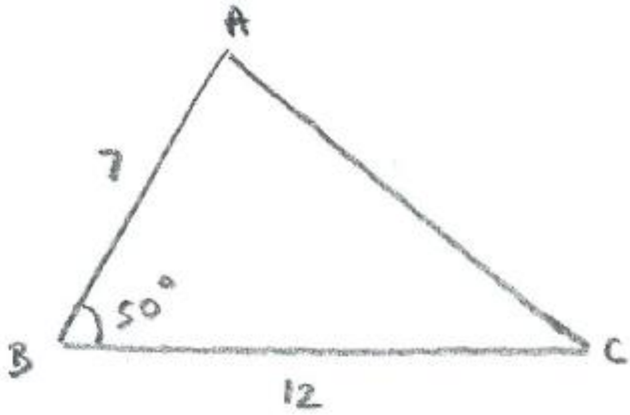
(III) a, b & C $(or\ b, c\ \&\ A\ etc)$ ⇒ unique solution

(IV) a, b, A $(or\ B)$ ⇒ 2 solutions in some cases (if A is acute, $a < b$ and $B \neq 90°$)

(ii) If we attempt to find an angle that is close to 90° using the Sine rule, then it will not be clear whether the correct angle is θ or $180 - \theta$.

To avoid this problem we can either use the Cosine rule, or apply the Sine rule only to angles that are clearly acute, and deduce the remaining angle by subtraction from 180°. Note that the angle opposite the longest side is the only one that can be obtuse.

(9) For the triangle below, what is the best strategy for finding angle A?



Solution

Although there is only one way of drawing the triangle, we want to avoid using $sinA$, as A is close to 90°

(and $sinx = \sin(180° - x)$)

But we could find C instead, and subtract $B + C$ from 180°.

We can use the Cosine rule to find AC, and then the Sine rule to find C, and hence A.

Alternatively, having found AC, we could use the Cosine rule again to find A (there is never any ambiguity when using the Cosine rule).

(10) If $sin\theta = 0.6$, where $0 \leq \theta < 360°$, find $tan\theta$

Solution

$tan\theta = \frac{sin\theta}{cos\theta} = \frac{sin\theta}{\pm\sqrt{1-sin^2\theta}} = \frac{\pm 0.6}{0.8} = \pm\frac{3}{4}$ (or draw graphs)

(11) Find expressions for $cos^2\theta$ & $sin^2\theta$ in terms of $cos2\theta$

Solution

First of all,

$cos^2\theta + sin^2\theta = 1$ and $cos^2\theta - sin^2\theta = cos2\theta$

Adding these gives $2cos^2\theta = 1 + cos2\theta$

and subtracting gives $2sin^2\theta = 1 - cos2\theta$

so that $cos^2\theta = \frac{1}{2}(1 + cos2\theta)$

and $sin^2\theta = \frac{1}{2}(1 - cos2\theta)$

(12) Assuming that $sin^2\theta + cos^2\theta = 1$, but without using any compound angle results, show that $sin\theta cos\theta \leq \frac{1}{2}$

Solution

$(sin\theta - cos\theta)^2 \geq 0 \Rightarrow sin^2\theta + cos^2\theta - 2sin\theta cos\theta \geq 0$

$\Rightarrow 1 \geq 2sin\theta cos\theta \Rightarrow sin\theta cos\theta \leq \frac{1}{2}$

(13) Show that $\frac{d}{d\phi}sin\phi = \frac{\pi}{180}cos\phi$, when ϕ is measured in degrees.

Solution

If ϕ is the angle in degrees, and θ is the angle in radians, so that $\phi = \left(\frac{180}{\pi}\right)\theta$, then

$$\frac{d}{d\phi} sin_{deg}\phi = \frac{d}{d\phi} sin_{rad}\theta = \left[\frac{d}{d\theta} sin_{rad}\theta\right]\frac{d\theta}{d\phi} = (cos_{rad}\theta)\left(\frac{\pi}{180}\right)$$

$$= (cos_{deg}\phi)\left(\frac{\pi}{180}\right)$$

[See "Trigonometry - Part 2" for alternative derivations, and related discussion.]