Transformations - Exercises (Sol'ns)(5 pages; 7/10/18)

(1) Suppose that we wish to reflect $y = f(x)$ in the line $x = a$. What combination of transformations could be used to do this?

Solution

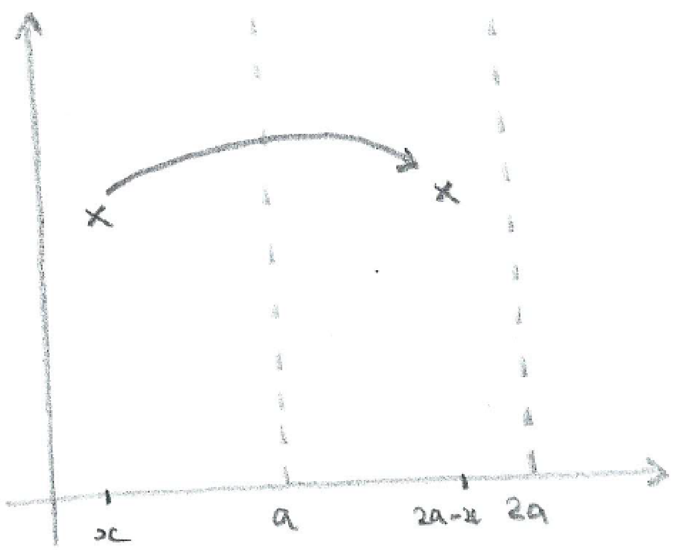
A particular point can be reflected in the line $x = a$ by considering a translation of a to the left, then performing a reflection in the y-axis and translating everything back, by a to the right.

In mathematical terms, x is first of all replaced by $x + a$; then x is replaced by $-x$, and finally x is replaced by $x - a$ (see note below).

Thus $f(x) \rightarrow f(x + a) \rightarrow f(-x + a) \rightarrow f(-[x - a] + a) = f(2a - x)$

[As an aid to memory, consider the reflection of $y = sinx$ about $x = \frac{\pi}{2}$, which is $y = \sin(\pi - x)$]

Alternative approach: $f(2a - x)$ can be justified by observing that when a point is reflected in the line $x = a$, its x coordinate changes from being x to the right of O (in the case where $x > 0$) to being x to the left of $2a$ (as in the example of $y = sinx$). See diagram below.



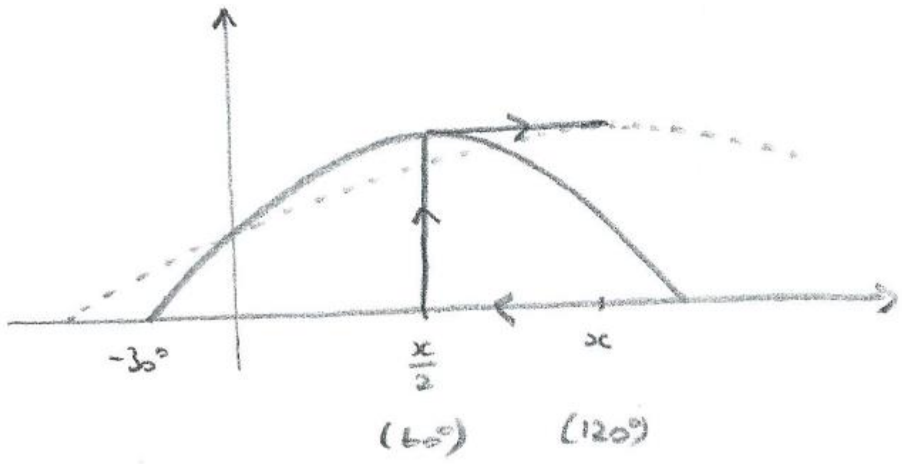
Note: An important point to observe when carrying out composite transformations is that, at any stage of the process, only the following operations are allowed: replacing x with $x+a$ (where a can be negative), or replacing x with kx (where k can be negative).

For example, if we want to stretch $y=\sin(x+30°)$ by a scale factor 2 in the x-direction, then the point $(x, \sin(x+30°)\ )$ is moving to $(2x, \sin(x+30°))$. Making the substitution $u=2x$, the coordinates of this point on the new curve are

$\left(u, \sin\left(\frac{u}{2}+30°\right)\right)$, and re-labelling, to give y as a function of x (rather than u), we have $y=\sin(\frac{x}{2}+30°)$.

Alternatively (going the other way): the graph of

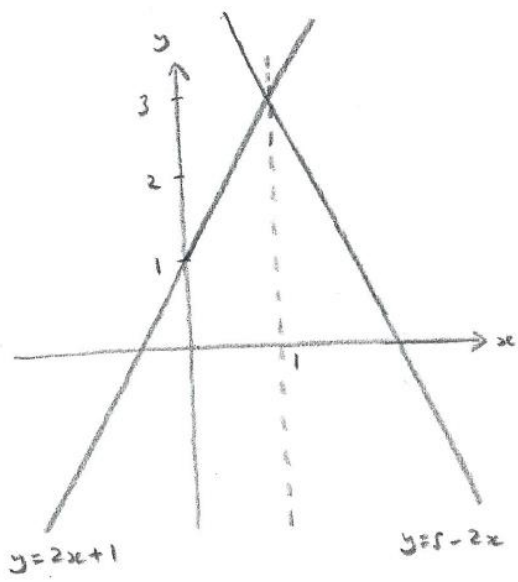
$y=\sin(\frac{x}{2}+30°)$ can be obtained as follows: we want the curve with coordinates $(x, \sin(\frac{x}{2}+30°)\ )$. This can be obtained from the curve with coordinates $(x, \sin(x+30°)\ )$ by 'looking to the left' of x, to find the point $\left(\frac{x}{2}, \sin\left(\frac{x}{2}+30°\right)\right)$, and then dragging it back to the right, to give $\left(x, \sin\left(\frac{x}{2}+30°\right)\right)$ [see diagram below]. (Note that, as this transformation is a stretch, the amount of dragging will depend on the distance from the Origin.) The dragging to the right explains why we see the curve stretching outwards (even though x is being replaced by $\frac{x}{2}$). A similar argument applies in the case of translations (though here the amount of dragging is the same for all points).



(2) Find the equation of the line resulting from the reflection of $y = 2x + 1$ in the line $x = 1$.

Solution

The transformed line is $y = 2(2 - x) + 1 = 5 - 2x$



Check: The transformed line will pass through the point where $y = 2x + 1$ meets the line $x = 1$; ie at $(1, 3)$, and will have a gradient of -2; hence its equation is $\frac{y-3}{x-1} = -2$ etc

(3) Describe the transformation represented by $y = e^x \to$
$y = e^{4-x}$

Solution

Step 1: Replace x with $-x$ (reflection in y-axis), to give $y = e^{-x}$

Step 2: Replace x with $x - 4$ (translation of $\binom{4}{0}$), to give

$y = e^{-(x-4)} = e^{4-x}$

So the transformation is a reflection in the y-axis, followed by a translation of $\binom{4}{0}$. This enables the graph to be sketched.
However, this compound transformation can be represented as a single transformation: in general, a reflection in the line $x = L$ is achieved by replacing x with $2L - x$, so that in this case we have a reflection in the line $x = 2$. [Consider the statement $\sin(\pi - \theta) = sin\theta$, which arises because of the symmetry of the sine curve about $\theta = \frac{\pi}{2}$.]

(4) What happens to the graph of $y = f(x)$ when it is transformed to:

(a) $y = f(|x|)$ (b) $|y| = f(x)$

Solution

(a) When $x \geq 0$, $f(|x|) = f(x)$; when $x < 0$, $f(|x|) = f(-x)$; ie that part of $y = f(x)$ to the right of the y-axis is reflected in the y-axis.

So $y = f(|x|)$ is the right half of $y = f(x)$, together with its reflection in the y-axis.

(b) First of all, $|y| = f(x)$ is only defined for x such that $f(x) \geq 0$.

The graph of $|y| = f(x)$ is similar to that of $y^2 = f(x)$, or

$y = \pm\sqrt{f(x)}$, in that it has two branches: $y = f(x)$ and

$y = -f(x)$.

So, provided $f(x) \geq 0$, $|y| = f(x)$ is the same as $y = f(x)$, with the addition of its reflection in the x-axis.

(5) What combination of transformations converts $y = 2^x$ to

$y = 2^{4x-2}$?

Solution

$y = 2^x \rightarrow y = 2^{4x}$ is a stretch of scale factor $\frac{1}{4}$ in the x-direction

Then $y = 2^{4x} \rightarrow y = 2^{4(x-\frac{1}{2})} = 2^{4x-2}$ is a translation of $\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$

[Alternatively, $y = 2^{4x} \rightarrow y = \left(\frac{1}{4}\right)2^{4x} = 2^{4x-2}$ is a stretch of scale factor $\frac{1}{4}$ in the y-direction.]

(6) Find the equation of the function resulting from a translation of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ of $y = 2x + 1$

Solution

$y = 2x + 1 \rightarrow y = [2(x - 1) + 1] + 2 = 2x + 1$