Shears - Exercises (Sol'ns) (4 pages; 22/2/20)

(1***) Consider the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, which represents a shear. Show that it is not possible for all of the elements of the matrix to be positive.

Solution

$ad - bc = 1$ & $a + d = 2$

$\Rightarrow a(2 - a) - bc = 1$

$\Rightarrow -bc = a^2 - 2a + 1 = (a - 1)^2$

If b & c are both positive, then $(a - 1)^2 < 0$, which isn't possible.

(2***) If the 2×2 matrix M represents a shear, what can be said about M^{-1}?

Solution

$|M^{-1}| = |M|$ and $\text{tr}(M^{-1}) = tr(M)$

[as $M^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$, if $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$] , so that M^{-1} will also represent a shear. It will be in the opposite direction to that represented by M.

(3***) Find the invariant lines of the shear represented by the matrix $\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$

Solution

The first step is to find the line of invariant points. This will be an eigenvector (passing through the Origin) with eigenvalue of 1.

$\begin{vmatrix} 4 - \lambda & -3 \\ 3 & -2 - \lambda \end{vmatrix} = 0 \Rightarrow (4 - \lambda)(-2 - \lambda) + 9 = 0$

$\Rightarrow \lambda^2 - 2\lambda + 1 = 0 \Rightarrow (\lambda - 1)^2 = 0 \Rightarrow \lambda = 1$

[This confirms that there is an eigenvalue of 1, but we could have skipped this step.]

$\begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow y = x$ is the line of invariant points

The invariant lines of the shear are the lines parallel to $y = x$;

ie $y = x + c$

Alternative method

$\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} x' \\ mx' + c \end{pmatrix}$ $\forall x$ [for all x]

$\Rightarrow 4x - 3mx - 3c = x'$ (1) & $3x - 2mx - 2c = mx' + c$ (2)

Substituting for x' from (1) into (2):

$x(3 - 2m) - 3c = m(4x - 3mx - 3c)$

$\Rightarrow x(3 - 2m - 4m + 3m^2) - 3c + 3mc = 0$

As this is to be true $\forall x$, we can equate powers of x, to give:

$3m^2 - 6m + 3 = 0$ and $-3c + 3mc = 0$;

ie $m^2 - 2m + 1 = 0$ and $c(m - 1) = 0$

so that $m = 1$ (and c can take any value),

and hence the invariant lines have the form $y = x + c$

(4***) For the shear $\begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}$, find the shear factor, and show that the point $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ is displaced by the expected amount in the direction of the line of shear.

Solution

From (4), the line of shear is $y = 2x$, and $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ is to the left of the line of shear, in the direction $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Hence the expected displacement in the direction of the line of shear is

$\frac{\boldsymbol{(a-1)^2+c^2}}{\boldsymbol{c}} \times$ perpendicular distance of $\begin{pmatrix}2\\5\end{pmatrix}$ from $y = 2x$

The eq'n of the line perpendicular to $y = 2x$ that passes through $\begin{pmatrix}2\\5\end{pmatrix}$ is $\frac{y-5}{x-2} = -\frac{1}{2}$, or $y = -\frac{1}{2}x + 6$

At the point of intersection of these two lines, $2x = -\frac{1}{2}x + 6$, so that $x = \frac{12}{5}$ and $y = \frac{24}{5}$

Hence the perpendicular distance is $\sqrt{(\frac{12}{5} - 2)^2 + (\frac{24}{5} - 5)^2}$

$= \sqrt{(\frac{2}{5})^2 + (\frac{-1}{5})^2} = \sqrt{\frac{1}{5}}$

So the expected displacement of $\begin{pmatrix}2\\5\end{pmatrix}$ is $\frac{(a-1)^2+c^2}{c} \times \sqrt{\frac{1}{5}}$

$= \frac{5}{\sqrt{5}} = \sqrt{5}$

and the expected image of $\begin{pmatrix}2\\5\end{pmatrix}$ is $\begin{pmatrix}2\\5\end{pmatrix} + \sqrt{5}\ \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\\7\end{pmatrix}$

[as $\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}$ is the unit vector in the direction of $\begin{pmatrix}1\\2\end{pmatrix}$]

And the image is fact $\begin{pmatrix}-1 & 1\\-4 & 3\end{pmatrix}\begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}3\\7\end{pmatrix}$

(5***) Show that $\frac{(a-1)^2+c^2}{c} = -\frac{b^2+(1-d)^2}{b}$ for the shear $\begin{pmatrix}a & c\\b & d\end{pmatrix}$.

Solution

$a + d = 2$ and $ad - bc = 1$

The required result is equivalent to

$b(a-1)^2 + bc^2 + cb^2 + c(1-d)^2 = 0$ (1)

As $1 - d = 1 - (2 - a) = a - 1$,

(1) $\Leftrightarrow (a-1)^2(b+c) + bc(b+c) = 0$

$\Leftrightarrow [(a-1)^2 + bc](b+c) = 0$

And $[(a-1)^2 + bc = (a-1)(1-d) + bc$

$= -(ad-bc) + (a+d) - 1$

$= -1 + 2 - 1 = 0$ as required.

(6***) Find the invariant lines of the shear represented by the matrix $\begin{pmatrix} 7 & -4 \\ 9 & -5 \end{pmatrix}$

Solution

Line of invariant points:

$\begin{pmatrix} 7 & -4 \\ 9 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow 7x - 4y = x \Rightarrow y = \frac{3x}{2}$

[this is the 'line of shear']

Invariant lines:

$\begin{pmatrix} 7 & -4 \\ 9 & -5 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} 7x - 4mx - 4c \\ 9x - 5mx - 5c \end{pmatrix}$

We require $9x - 5mx - 5c = m(7x - 4mx - 4c) + c$ (for all x)

Equating coefficients of x, $9 - 5m = 7m - 4m^2$,

so that $4m^2 - 12m + 9 = 0$

$\Rightarrow (2m-3)^2 = 0 \Rightarrow m = \frac{3}{2}$ (1)

Equating constant terms: $-5c = -4mc + c$

$\Rightarrow 0 = c(6 - 4m) \Rightarrow c = 0 \; or \; m = \frac{3}{2}$ (2)

In order for both (1) & (2) to hold, $m = \frac{3}{2}$

ie the invariant lines are $y = \frac{3x}{2} + c$

(parallel to the line of shear)