

## Shears - Exercises (Sol'ns) (4 pages; 4/6/19)

(1) Consider the matrix  $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ , which represents a shear. Show that it is not possible for all of the elements of the matrix to be positive.

### Solution

$$ad - bc = 1 \quad \& \quad a + d = 2$$

$$\Rightarrow a(2 - a) - bc = 1$$

$$\Rightarrow -bc = a^2 - 2a + 1 = (a - 1)^2$$

If  $b$  &  $c$  are both positive, then  $(a - 1)^2 < 0$ , which isn't possible.

(2) If the  $2 \times 2$  matrix  $M$  represents a shear, what can be said about  $M^{-1}$ ?

### Solution

$$|M^{-1}| = |M| \quad \text{and} \quad \text{tr}(M^{-1}) = \text{tr}(M)$$

[as  $M^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ , if  $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ ], so that  $M^{-1}$  will also represent a shear. It will be in the opposite direction to that represented by  $M$ .

(3) Find the invariant lines of the shear represented by the matrix  $\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$

### Solution

The first step is to find the line of invariant points. This will be an eigenvector (passing through the Origin) with eigenvalue of 1.

$$\begin{vmatrix} 4 - \lambda & -3 \\ 3 & -2 - \lambda \end{vmatrix} = 0 \Rightarrow (4 - \lambda)(-2 - \lambda) + 9 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 = 0 \Rightarrow (\lambda - 1)^2 = 0 \Rightarrow \lambda = 1$$

[This confirms that there is an eigenvalue of 1, but we could have skipped this step.]

$$\begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow y = x \text{ is the line of invariant points}$$

The invariant lines of the shear are the lines parallel to  $y = x$ ;

ie  $y = x + c$

### Alternative method

$$\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} x' \\ mx' + c \end{pmatrix} \forall x \text{ [for all } x]$$

$$\Rightarrow 4x - 3mx - 3c = x' \text{ (1) \& } 3x - 2mx - 2c = mx' + c \text{ (2)}$$

Substituting for  $x'$  from (1) into (2):

$$x(3 - 2m) - 3c = m(4x - 3mx - 3c)$$

$$\Rightarrow x(3 - 2m - 4m + 3m^2) - 3c + 3mc = 0$$

As this is to be true  $\forall x$ , we can equate powers of  $x$ , to give:

$$3m^2 - 6m + 3 = 0 \text{ and } -3c + 3mc = 0;$$

$$\text{ie } m^2 - 2m + 1 = 0 \text{ and } c(m - 1) = 0$$

so that  $m = 1$  (and  $c$  can take any value),

and hence the invariant lines have the form  $y = x + c$

(4) For the shear  $\begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}$ , find the shear factor, and show that the point  $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$  is displaced by the expected amount in the direction of the line of shear.

### Solution

From (4), the line of shear is  $y = 2x$ , and  $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$  is to the left of the line of shear, in the direction  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ . Hence the expected displacement in the direction of the line of shear is

$$\frac{(a-1)^2+c^2}{c} \times \text{perpendicular distance of } \begin{pmatrix} 2 \\ 5 \end{pmatrix} \text{ from } y = 2x$$

The eq'n of the line perpendicular to  $y = 2x$  that passes through  $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$  is  $\frac{y-5}{x-2} = -\frac{1}{2}$ , or  $y = -\frac{1}{2}x + 6$

At the point of intersection of these two lines,  $2x = -\frac{1}{2}x + 6$ , so that  $x = \frac{12}{5}$  and  $y = \frac{24}{5}$

$$\begin{aligned} \text{Hence the perpendicular distance is } & \sqrt{\left(\frac{12}{5} - 2\right)^2 + \left(\frac{24}{5} - 5\right)^2} \\ & = \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{-1}{5}\right)^2} = \sqrt{\frac{1}{5}} \end{aligned}$$

$$\begin{aligned} \text{So the expected displacement of } \begin{pmatrix} 2 \\ 5 \end{pmatrix} \text{ is } & \frac{(a-1)^2+c^2}{c} \times \sqrt{\frac{1}{5}} \\ & = \frac{5}{\sqrt{5}} = \sqrt{5} \end{aligned}$$

$$\text{and the expected image of } \begin{pmatrix} 2 \\ 5 \end{pmatrix} \text{ is } \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \sqrt{5} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

[as  $\frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  is the unit vector in the direction of  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ]

$$\text{And the image is fact } \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

$$(5) \text{ Show that } \frac{(a-1)^2+c^2}{c} = -\frac{b^2+(1-d)^2}{b} \text{ for the shear } \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

**Solution**

$$a + d = 2 \text{ and } ad - bc = 1$$

The required result is equivalent to

$$b(a-1)^2 + bc^2 + cb^2 + c(1-d)^2 = 0 \quad (1)$$

$$\text{As } 1-d = 1-(2-a) = a-1,$$

$$(1) \Leftrightarrow (a-1)^2(b+c) + bc(b+c) = 0$$

$$\Leftrightarrow [(a-1)^2 + bc](b+c) = 0$$

$$\begin{aligned} \text{And } [(a-1)^2 + bc] &= (a-1)(1-d) + bc \\ &= -(ad - bc) + (a+d) - 1 \\ &= -1 + 2 - 1 = 0 \text{ as required.} \end{aligned}$$

(6) Find the invariant lines of the shear represented by the matrix

$$\begin{pmatrix} 7 & -4 \\ 9 & -5 \end{pmatrix}$$

### Solution

Line of invariant points:

$$\begin{pmatrix} 7 & -4 \\ 9 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow 7x - 4y = x \Rightarrow y = \frac{3x}{2}$$

[this is the 'line of shear']

Invariant lines:

$$\begin{pmatrix} 7 & -4 \\ 9 & -5 \end{pmatrix} \begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} 7x - 4mx - 4c \\ 9x - 5mx - 5c \end{pmatrix}$$

We require  $9x - 5mx - 5c = m(7x - 4mx - 4c) + c$  (for all  $x$ )

Equating coefficients of  $x$ ,  $9 - 5m = 7m - 4m^2$ ,

so that  $4m^2 - 12m + 9 = 0$

$$\Rightarrow (2m - 3)^2 = 0 \Rightarrow m = \frac{3}{2} \quad (1)$$

Equating constant terms:  $-5c = -4mc + c$

$$\Rightarrow 0 = c(6 - 4m) \Rightarrow c = 0 \text{ or } m = \frac{3}{2} \quad (2)$$

In order for both (1) & (2) to hold,  $m = \frac{3}{2}$

ie the invariant lines are  $y = \frac{3x}{2} + c$

(parallel to the line of shear)