Series – Q6 [Practice/H] (20/6/21)

By considering $\sum_{r=1}^{n}(r+1)^4$, show that

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ [The results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ can be assumed.]

Solution

$$\sum_{r=1}^{n}(r+1)^4 = \left(\sum_{r=1}^{n} r^4\right) + 4\left(\sum_{r=1}^{n} r^3\right) + 6\left(\sum_{r=1}^{n} r^2\right) + 4\left(\sum_{r=1}^{n} r\right) + n$$

Let $R = r + 1$, so that LHS $= \sum_{R=2}^{n+1} R^4 = (\sum_{R=1}^{n} R^4) + (n+1)^4 - 1$

Thus $(\sum_{R=1}^{n} R^4) + (n+1)^4 - 1 = (\sum_{r=1}^{n} r^4) + 4(\sum_{r=1}^{n} r^3) + 6(\sum_{r=1}^{n} r^2) + 4(\sum_{r=1}^{n} r) + n$

and so $(n+1)^4 - 1 = 4(\sum_{r=1}^{n} r^3) + 6(\sum_{r=1}^{n} r^2) + 4(\sum_{r=1}^{n} r) + n$

Hence $4(\sum_{r=1}^{n} r^3) = n^4 + 4n^3 + 6n^2 + 4n - n(n+1)(2n+1) - 2n(n+1) - n$

$= n^4 + 4n^3 + 6n^2 + 4n - (2n^3 + 3n^2 + n) - 2n^2 - 2n - n$

$= n^4 + 2n^3 + n^2 = n^2(n^2 + 2n + 1) = n^2(n+1)^2$

and $\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2(n+1)^2$