

Scalar Product (2 pages; 4/8/18)

(1) The scalar (or 'dot') product of the vectors \underline{a} and \underline{b} (which can be 2D or 3D) is defined as:

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|\cos\theta, \text{ where } \theta \text{ is the angle between } \underline{a} \text{ and } \underline{b}.$$

Note that $\underline{b} \cdot \underline{a} = \underline{a} \cdot \underline{b}$ (as $\cos(-\theta) = \cos\theta$)

$$(2) \underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1$$

$$\text{and } \underline{i} \cdot \underline{j} = \underline{j} \cdot \underline{k} = \underline{k} \cdot \underline{i} = 0$$

$$\text{So } (a_1\underline{i} + a_2\underline{j} + a_3\underline{k}) \cdot (b_1\underline{i} + b_2\underline{j} + b_3\underline{k}) = a_1b_1 + a_2b_2 + a_3b_3$$

$$\text{Hence } a_1b_1 + a_2b_2 + a_3b_3 = |\underline{a}||\underline{b}|\cos\theta$$

(3) $\underline{a} \cdot \underline{b} = 0 \Rightarrow \underline{a}$ and \underline{b} are perpendicular (assuming that they aren't zero)

(4) One application of the scalar product is in determining the work done (W) by a force \underline{F} , which has component $|\underline{F}|\cos\theta$ in the direction of the displacement \underline{s} of the object in question, so that

$$W = \underline{F} \cdot \underline{s}$$

(5) Example

A triangle is made up of the points $O(0,0)$, $A(-3,4)$ and $B(12,5)$.

Find the angle between the sides OA and OB

$$\cos\theta = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|}$$

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 5 \end{pmatrix} = (-3)(12) + (4)(5) = -16$$

$$|\overrightarrow{OA}| = \sqrt{(-3)^2 + 4^2} = 5 \quad \& \quad |\overrightarrow{OB}| = \sqrt{12^2 + 5^2} = 13$$

$$\text{Hence } \cos\theta = \frac{-16}{(5)(13)} = -0.24615 \quad \text{and } \theta = 104.2^\circ$$

[Note: The scalar product often provides a more convenient alternative to using the Cosine rule.]

(6) Useful devices

$$(i) \underline{a} \cdot \underline{a} = |\underline{a}|^2$$

$$(ii) \text{ If } \underline{r} \cdot \underline{a} = 0 \text{ for all } \underline{r}, \text{ then } \underline{a} = 0$$

(7) Consider a parallelogram with sides \underline{u} & \underline{v} . Then, if the diagonals of the parallelogram are perpendicular,

$$(\underline{u} + \underline{v}) \cdot (\underline{u} - \underline{v}) = 0$$

Expanding this gives $\underline{u} \cdot \underline{u} = \underline{v} \cdot \underline{v}$; ie $|\underline{u}|^2 = |\underline{v}|^2$, so that the two sides are equal; ie the parallelogram is a rhombus.