

**STEP - Transformations** (6 pages; 25/2/18)

(1) Translation of  $\begin{pmatrix} a \\ b \end{pmatrix}$ :  $y = f(x) \rightarrow y - b = f(x - a)$

(2) Stretch of scale factor  $k$  in the  $x$  direction:

$$y = f(x) \rightarrow y = f\left(\frac{x}{k}\right)$$

Stretch of scale factor  $k$  in the  $y$  direction:  $y = f(x) \rightarrow \frac{y}{k} = f(x)$

(3) Note that, at each stage of a composite transformation, we must be replacing  $x$  by either  $x + a$  (where  $a$  can be negative) or  $kx$  (and similarly for  $y$ ).

(4) Reflection in the line  $x = L$ :  $y = f(x) \rightarrow y = f(2L - x)$

Reflection in the line  $y = L$ :  $y = f(x) \rightarrow 2L - y = f(x)$

Special cases:

Reflection in the line  $x = 0$ :  $f(x) \rightarrow f(-x)$

Reflection in the line  $y = 0$ :  $y = f(x) \rightarrow -y = f(x)$

See also Appendix.

(5) **Example:** Applying stretches of scale factors  $a$  &  $b$  in the  $x$  &  $y$  directions to the circle  $x^2 + y^2 = 1$  gives the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

(6) **Example:** To obtain  $y = \sin(2x + 60)$  from  $y = \sin x$ ,

**either** (a) stretch by scale factor  $\frac{1}{2}$  in the  $x$  direction, to give

$y = \sin(2x)$ , and then translate by  $\begin{pmatrix} -30 \\ 0 \end{pmatrix}$ , to give

$$y = \sin(2[x + 30]) = \sin(2x + 60)$$

**or** (b) translate by  $\begin{pmatrix} -60 \\ 0 \end{pmatrix}$ , to give  $y = \sin(x + 60)$ , and then

stretch by scale factor  $\frac{1}{2}$  in the  $x$  direction, to give

$y = \sin(2x + 60)$  [It is perhaps more awkward to produce a sketch by method (b).]

[Note that, at each stage, we are either replacing  $x$  by  $kx$ , or by  $x \pm a$ ]

(7) Transformations involving the modulus

(i)  $y = f(|x|)$

$f(|x|) = f(x)$  when  $x \geq 0$

$f(|x|) = f(-x)$  when  $x < 0$  (ie the left-hand half of  $y = f(x)$  is replaced by the reflection of the right-hand half in the  $y$ -axis)

(ii)  $|y| = f(x)$

As  $|y| \geq 0$ , the graph is undefined where  $f(x) < 0$ .

Where  $f(x) \geq 0$ , the graph of  $|y| = f(x)$  is that of  $y = f(x)$ , together with its reflection in the  $x$ -axis.

## (8) Rotation about general point

A rotation of  $\theta^\circ$  [anti-clockwise] (of eg a shape) about a point  $(a,b)$  is equivalent to a translation  $\begin{pmatrix} -a \\ -b \end{pmatrix}$ , followed by a rotation of  $\theta^\circ$  about the origin, and a translation of  $\begin{pmatrix} a \\ b \end{pmatrix}$ . (\*)

When  $\theta = 180$ , another equivalent combination of transformations is a reflection in the line  $x = a$ , followed by a reflection in the line  $y = b$ . (\*\*)

**Exercise:** Demonstrate the equivalence of (\*) and (\*\*) for the general function  $y = f(x)$  (when  $\theta = 180$ ).

**Solution**

For (\*):

$$y = f(x) \rightarrow y + b = f(x + a) \text{ [translation of } \begin{pmatrix} -a \\ -b \end{pmatrix}]$$

$$\rightarrow -y + b = f(-x + a) \text{ [reflection in } x = 0 \text{ \& reflection in } y = 0;$$

equivalent to rotation of  $180^\circ$  about the origin]

$$\rightarrow -(y - b) + b = f(-[x - a] + a); \text{ ie } 2b - y = f(2a - x)$$

For (\*\*):

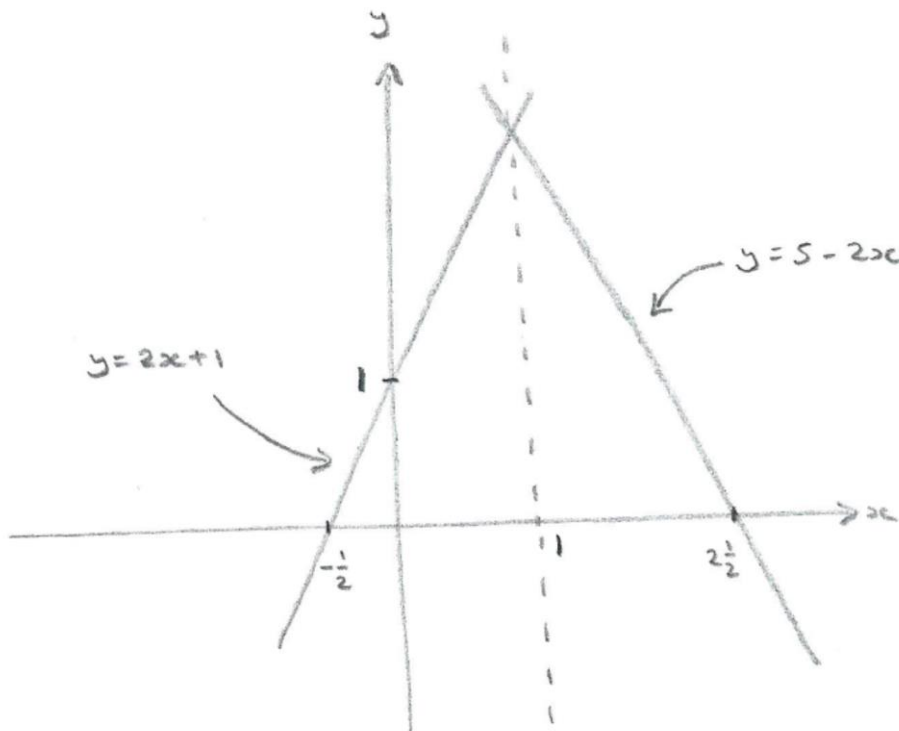
For a reflection in the line  $x = a$ ,  $x$  is replaced by  $2a - x$  (see separate note on reflection in line  $x = L$ ). Similarly, for a reflection in  $y = b$ ; giving  $2b - y = f(2a - x)$ , as above.

## Appendix: Reflection in line $x = L$ : Example & Exercises

**Example:** Reflect the line  $y = 2x + 1$  in the line  $x = 1$

### Method A

First reflect  $y = 2x + 1$  in the line  $x = 0$ . The image intersects the line  $y = 2x + 1$  at  $(0,1)$ . For a reflection in the line  $x = 1$ , we want the image of  $(0,1)$  to be at  $(2,1)$ . So a translation of  $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$  is required.



Algebraically,  $y = 2x + 1$  is first transformed to

$$y = 2(-x) + 1 = 1 - 2x;$$

$$\text{then to } y = 1 - 2(x - 2) = 5 - 2x$$

## Method B

The new line will have gradient  $-2$ , and meet the line  $y = 2x + 1$  when  $x = 1$ ; ie at the point  $(1, 3)$ .

So it has equation  $\frac{y-3}{x-1} = -2$  ; ie  $y - 3 = -2x + 2$ , or  $y = 5 - 2x$

(obviously method A can be applied more generally than B)

## Method C

In order to reflect the curve  $y = f(x)$  in the line  $x = L$ , to give the function  $y = g(x)$ , we require  $g(L + u) = f(L - u)$

Let  $x = L + u$ , so that  $L - u = L - (x - L) = 2L - x$

Thus, for a reflection in the line  $x = L$ ,  $f(x)$  is transformed to  $f(2L - x)$ .

This is equivalent to a reflection in  $x = 0$ , followed by a translation of  $\begin{pmatrix} 2L \\ 0 \end{pmatrix}$ :  $f(x) \rightarrow f(-x) \rightarrow f(-[x - 2L]) = f(2L - x)$ .

**Note:** The result  $\sin(\pi - x) = \sin x$  follows from the fact that  $y = \sin x$  is symmetrical about  $x = \frac{\pi}{2}$ .

**Exercise 1:** Find the function resulting from reflecting  $y = \cos x$  in the line  $x = \frac{\pi}{2}$ , and then in the line  $y = 1$

**Solution**

For a reflection in the line  $x = \frac{\pi}{2}$ , first reflect in  $x = 0$  :

$$y = \cos x \rightarrow y = \cos(-x) = \cos x;$$

then translate by  $\begin{pmatrix} \pi \\ 0 \end{pmatrix} \rightarrow y = \cos(x - \pi) = \cos x \cos \pi + \sin x \sin \pi;$

$$\text{ie } y = -\cos x$$

Then for a reflection in the line  $y = 1$ , first reflect in  $y = 0$  :

$$y = -\cos x \rightarrow -y = -\cos x; \text{ ie } y = \cos x ;$$

and then translate by  $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ , to give  $y - 2 = \cos x;$

$$\text{ie } y = \cos x + 2$$

**Exercise 2**

Symmetry in  $x = L$  occurs when  $f(L - x) = f(L + x)$

Show that this is equivalent to  $f(2L - x) = f(x)$ .

**Solution**

If  $f(L - x) = f(L + x)$ , then

$$f(2L - x) = f(L + [L - x]) = f(L - [L - x]) = f(x)$$

Conversely, if  $f(2L - x) = f(x)$ , then

$$f(L - x) = f(2L - L - x) = f(2L - [L + x]) = f(L + x)$$