STEP Topics - Notes (5 pages; 25/3/18)

Algebra Methods: see separate document

Complex Numbers: see separate document

Counting

(1) Case by case approach

- provided there aren't too many cases

(2) Look ahead in the question: sometimes a method needs to be extended, and may be too unwieldy for the next part of the question.

(3) Total number of items - number of unwanted items

Curve Sketching: see separate document

Differential Equations: see separate document

Induction

See Pure:"Induction" & Pure/Exercises:"Induction"

(1) Types of examples:

(a) $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$

(b) If $u_n = 3u_{n-1} + 4$, where $u_1 = 2$, then $u_n = 4(3^{n-1}) - 2$

(c) $7^{2n-1} + 3^{2n}$ is divisible by 8

(d) $\sum_{r=1}^{n} r^2 > \frac{1}{3}n^3$

(e) The sum of the interior angles of a convex n-sided polygon is $180(n-2)$

(f) $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ for $n \geq 2$

(g) If $y = e^x sinx$, show that $\frac{d^n y}{dx^n} = \left(\sqrt{2}\right)^n e^x \sin(x + \frac{n\pi}{4})$

(2) 'Weak' and 'strong' induction

[$P(k)$ is the proposition that a particular result is true for $n = k$]

'Weak' induction is just the ordinary method; 'strong' induction is where we show that if $P(k-m), P(k-m+1), \ldots P(k)$ are correct, then $P(k+1)$ will be correct. We then have to establish that $P(1), P(2), \ldots P(m+1)$ are correct. [In some cases we might start at eg $P(0)$.]

Example of strong induction

g_n is defined recursively as $(n^3 - 3n^2 + 2n)g_{n-3}$ for $n \geq 4$, and $g_1 = 1, g_2 = 2, g_3 = 6$

Show that $g_n = n!$ for $n \geq 1$

Proof

Assume that the result is true for $n = k-2, k-1$ & k.

Then $g_{k+1} = \left((k+1)^3 - 3(k+1)^2 + 2(k+1)\right)g_{k-2}$

$= (k^3 + 3k^2 + 3k + 1 - 3k^2 - 6k - 3 + 2k + 2)(k-2)!$

$= (k^3 - k)(k-2)!$

$= k(k-1)(k+1)(k-2)!$

$= (k+1)!$

So that the result is true for $n = k+1$ if it is true for

$n = k-2, k-1$ & k.

As it is true for $n = 1, 2$ & 3, it is therefore true for $n = 4, 5$, ... , and hence, by the principle of induction, for all positive integers.

Inequalities

See: Pure/Exercises:"Inequalities"

(1) If an expression can be arranged into the form $(a \pm b)^2$, then this will be non-negative.

(2) Consider critical values where equality holds.

(3) $a < b \Rightarrow \frac{1}{a} + \frac{1}{b} < \frac{1}{a-\delta} + \frac{1}{b+\delta}$ $(\delta > 0)$

(4) Possible use of linear interpolation, to obtain lower or upper bound.

(5) Beware of multiplying inequalities by a quantity that is (or could be) negative (eg log(0.5)).

Integers

(1) Case by case: n odd or even; $0, 1$ or 2 mod 3 (eg)

(2) Factorisation

eg to find possible integer solutions (for x & y) of $f(x, y) = c$, rearrange into the form $g(x)h(y) = d$, where d is an integer; there will now only be certain values that $g(x)$ & $h(y)$ can take

(3) Pairs of numbers x, y might be represented by coordinates (x, y); eg to find possible values of x & y, determine the number of grid points within the relevant area.

(4) Test for divisibility by 11

eg $11 \times 325847 = 3584317$

$3 - 5 + 8 - 4 + 3 - 1 + 7 = 11$, which is a multiple of 11, so 3584317 is a multiple of 11

(5) Difference of 2 squares; eg $(n+1)^2 - n^2 = 2n + 1$

Integration

See: Pure: "Integration Methods"

& Pure/Exercises:"Integration - Part 2"

Logarithms

See: Pure: "Logarithms"

Logarithms don't feature that much in STEP papers.

Probability

See separate note

Proof

(1) As an alternative to proving that $A \Rightarrow B\ and\ B \Rightarrow A$, it may be easier to prove that $A \Rightarrow B\ and\ A' \Rightarrow B'$ (as $A' \Rightarrow B'$ is equivalent to $B \Rightarrow A$

(2) Beware of using ⇒ when ⇔ is required.

(3) Simplify problem by WLOG ("without loss of generality").

(4) Consider the range of values for which a result is valid (eg if a binomial expansion is involved).

Trigonometry

See: Pure:"Trigonometry - Parts 1 & 2"

Vectors: see separate document