STEP Problems - Trigonometry (Sol'ns) (4 pages; 7/9/18)

(1) Solve $sin\theta = cos4\theta$ for $0 < \theta < \pi$

Solution

$sin\theta = sin(\frac{\pi}{2} - 4\theta)$

Hence $\theta = \frac{\pi}{2} - 4\theta + 2n\pi$ (1) or $\theta = \left(\pi - \left[\frac{\pi}{2} - 4\theta\right]\right) + 2n\pi$ (2)

From (1), $5\theta = \frac{\pi(1+4n)}{2}$, so that $\theta = \frac{\pi(1+4n)}{10}$

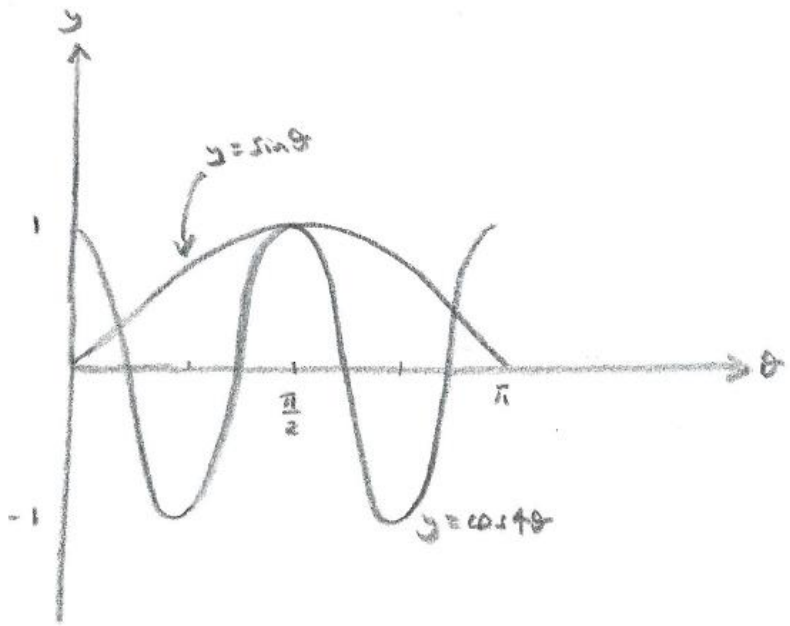
giving $\theta = \frac{\pi}{10}$, $\frac{\pi}{2}$ or $\frac{9\pi}{10}$

From (2), $-3\theta = \frac{\pi(1+4n)}{2}$, so that $\theta = \frac{-\pi(1+4n)}{6}$

giving $\theta = \frac{\pi}{2}$ again

Thus, the solutions are $\theta = \frac{\pi}{10}$, $\frac{\pi}{2}$ or $\frac{9\pi}{10}$

A sketch confirms that these are plausible.



Note: We could attempt $sin\theta = 1 - 2sin^2(2\theta)$

$= 1 - 8sin^2\theta(1 - sin^2\theta)$

giving $8x^4 - 8x^2 - x + 1 = 0$, where $x = sin\theta$

and we can see that $x = 1$ is a root, by the Factor theorem (or by spotting that $\theta = \frac{\pi}{2}$ satisfies the original equation).

However, even if we could solve the remaining cubic, we couldn't guarantee to be able to obtain θ from $x = sin\theta$ without a calculator.

(2) Given that $cos^5\theta = \frac{1}{16}(cos5\theta + 5cos3\theta + 10cos\theta)$ and

$cos^6\theta = \frac{1}{32}(cos6\theta + 6cos4\theta + 15cos2\theta + 10)$,

find expressions for $sin^5\theta$ and $sin^6\theta$

Solution

$sin^5\theta = cos^5\left(\frac{\pi}{2} - \theta\right)$

$= \frac{1}{16}(\cos[5\left(\frac{\pi}{2} - \theta\right)] + 5\cos[3(\frac{\pi}{2} - \theta)] + 10\cos(\frac{\pi}{2} - \theta))$

$= \frac{1}{16}(\cos[\frac{\pi}{2} - 5\theta] + 5\cos[-\frac{\pi}{2} - 3\theta] + 10\sin\theta)$

$= \frac{1}{16}(\sin 5\theta + 5\cos(\frac{\pi}{2} + 3\theta) + 10\sin\theta)$

$= \frac{1}{16}(\sin5\theta + 5\cos(\frac{\pi}{2} - [-3\theta]) + 10\sin\theta)$

$= \frac{1}{16}(sin5\theta + 5\sin(-3\theta) + 10sin\theta)$

$= \frac{1}{16}(sin5\theta - 5\sin3\theta + 10sin\theta)$

And $sin^6\theta = cos^6\left(\frac{\pi}{2} - \theta\right)$

$= \frac{1}{32}(\cos[6\left(\frac{\pi}{2} - \theta\right)] + 6\cos[4\left(\frac{\pi}{2} - \theta\right)] + 15\cos[2\left(\frac{\pi}{2} - \theta\right)] + 10)$

$= \frac{1}{32}(\cos(\pi - 6\theta) + 6\cos(-4\theta) + 15\cos(\pi - 2\theta) + 10)$

$= \frac{1}{32}(-cos6\theta + 6cos4\theta - 15cos2\theta + 10)$

(3) Simplify $\sqrt{2(1 - cos\theta)}$ and $\sqrt{2(1 + cos\theta)}$

Solution

$\cos\theta = cos^2(\theta/2)$ - $sin^2(\theta/2) = 1$ - $2sin^2(\theta/2)$

so that $1 - \cos\theta = 2sin^2(\theta/2)$

and $\sqrt{2(1 - cos\theta)} = 2\sin(\theta/2)$

Also, $\cos 2\theta = cos^2\theta$ - $sin^2\theta = 2cos^2\theta - 1$, so that $1 + \cos\theta = 2cos^2(\frac{\theta}{2})$

and $\sqrt{2(1 + cos\theta)} = 2\cos(\frac{\theta}{2})$

(4) Assuming that $sin^2\theta + cos^2\theta = 1$, but without using any compound angle results, show that $sin\theta cos\theta \leq \frac{1}{2}$

Solution

$(sin\theta - cos\theta)^2 \geq 0 \Rightarrow sin^2\theta + cos^2\theta - 2sin\theta cos\theta \geq 0$

$\Rightarrow 1 \geq 2sin\theta cos\theta \Rightarrow sin\theta cos\theta \leq \frac{1}{2}$

(5) Show that $\arctan\left(\frac{1+a}{\sqrt{1-a^2}}\right) - \arctan\left(\frac{a}{\sqrt{1-a^2}}\right) = \arctan\left(\frac{\sqrt{1-a}}{\sqrt{1+a}}\right)$

Solution

Consider $\theta - \phi$, where $tan\theta = A$ & $tan\phi = B$

Then $tan(\theta - \phi) = \frac{A-B}{1+AB}$

With $A = \frac{1+a}{\sqrt{1-a^2}}$ and $B = \frac{a}{\sqrt{1-a^2}}$,

$\arctan\left(\frac{1+a}{\sqrt{1-a^2}}\right) - \arctan\left(\frac{a}{\sqrt{1-a^2}}\right) = \theta - \phi = arctan(\frac{A-B}{1+AB})$

and $\frac{A-B}{1+AB} = \frac{[(1+a)-a]\sqrt{1-a^2}}{(1-a^2)+(1+a)a} = \frac{\sqrt{1-a^2}}{1+a} = \frac{\sqrt{1-a}\sqrt{1+a}}{1+a} = \frac{\sqrt{1-a}}{\sqrt{1+a}}$,

giving the required result.

(6) What is the period of $2\,sin\left(3x + \frac{\pi}{4}\right) + 3cos(\frac{2x}{3} - \frac{\pi}{3})$?

Solution

The period T_1 of $2\,sin\left(3x + \frac{\pi}{4}\right)$ satisfies $3T_1 = 2\pi$

[as $2sin\left(3[0] + \frac{\pi}{4}\right) = 2sin\left(2\pi + \frac{\pi}{4}\right)$]; ie $T_1 = \frac{2\pi}{3}$

Similarly for $3cos(\frac{2x}{3} - \frac{\pi}{3})$, $\frac{2T_2}{3} = 2\pi$, so that $T_2 = 3\pi$

The period of the sum of these functions is the LCM of these two periods; ie 6π.