STEP Problems - Algebra (Sol'ns) (6 pages; 6/9/18)

(1) Show that $\frac{sec\theta+1-tan\theta}{sec\theta+1+tan\theta} \equiv sec\theta - tan\theta$

Solution

Method 1

To show that $\frac{sec\theta+1-tan\theta}{sec\theta+1+tan\theta} - (sec\theta - tan\theta) \equiv 0$:

LHS $= \frac{(sec\theta+1-tan\theta)-(sec\theta-tan\theta)(sec\theta+1+tan\theta)}{sec\theta+1+tan\theta}$

Numerator $= (sec\theta + 1 - tan\theta)$

$-(sec\theta - tan\theta)(sec\theta + tan\theta) - (sec\theta - tan\theta)$

$= (sec\theta + 1 - tan\theta) - (sec^2\theta - tan^2\theta) - (sec\theta - tan\theta)$

$= (sec\theta + 1 - tan\theta) - 1 - (sec\theta - tan\theta) = 0$, as required.

Method 2

Force the LHS into the form $(sec\theta - tan\theta)f(\theta)$, and show that $f(\theta) \equiv 1$:

$\frac{sec\theta+1-tan\theta}{sec\theta+1+tan\theta} = (sec\theta - tan\theta).\frac{(sec\theta+1-tan\theta)}{(sec\theta-tan\theta)(sec\theta+1+tan\theta)}$

$= (sec\theta - tan\theta)f(\theta)$, say

Note that $(sec\theta - tan\theta)(sec\theta + tan\theta) = sec^2\theta - tan^2\theta = 1$

Thus the denominator of $f(\theta)$ equals $1 + (sec\theta - tan\theta)$, and hence $f(\theta) = 1$.

(2) Solve the equation $x - \sqrt{x} = 6$

Solution

Let $f(x) = x - \sqrt{x} - 6$

$f(x) = 0 \Rightarrow x - 6 = \sqrt{x}$

$\Rightarrow (x-6)^2 = x$, but this may include spurious solutions

[of $x - 6 = -\sqrt{x}$]

$\Rightarrow x^2 - 13x + 36 = 0$

$\Rightarrow (x-9)(x-4) = 0$

$\Rightarrow x = 9 \; or \; x = 4$

$f(9) = 0$ & $f(4) = -4$

Thus the only solution is $x = 9$

[Let $g(x) = x + \sqrt{x} - 6 = 0$

Then $g(x) = 0 \Rightarrow (x-6)^2 = x$ as well

$g(9) \neq 0, and \; g(4) = 0$]

Alternatively: Let $y = \sqrt{x}$, so that

$x - \sqrt{x} - 6 = 0 \Rightarrow y^2 - y - 6 = 0$

$\Rightarrow (y+2)(y-3) = 0$

$\Rightarrow y = -2$ (reject as $\sqrt{x}$ must be ≥ 0) $or \; y = 3$

(3) Given that $\frac{bc-a}{1-c} = 7$ & $\frac{b^2c-a^2}{1-c} = 51$, show that $\frac{a+7}{a^2+51} = \frac{b+7}{b^2+51}$

Solution

$\frac{bc-a}{1-c} = 7 \quad \Rightarrow bc - a = 7 - 7c \Rightarrow c(b+7) = 7 + a$

$\Rightarrow c = \frac{a+7}{b+7}$

and replacing a, b & 7 with a^2, b^2 & 51 gives $c = \frac{a^2+51}{b^2+51}$

so that $\frac{a+7}{b+7} = \frac{a^2+51}{b^2+51}$ and hence $\frac{a+7}{a^2+51} = \frac{b+7}{b^2+51}$ (since $a^2 + 51$ & $b^2 + 51$ are both non-zero)

(4) Express the following parametric equations in Cartesian form (ie a relation between x & y).

(i) $x = 2t + t^2,\ y = 2t^2 + t^3$

(ii) $x = 5t^2 - 4,\ y = 9t - t^3$

Solution

(i) $x = 2t + t^2,\ y = 2t^2 + t^3 \Rightarrow x = t(2 + t), y = t^2(2 + t)$

So $\frac{y}{x} = t$; then $x = 2\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2$ and hence $x^3 = 2xy + y^2$

(ii) $x = 5t^2 - 4,\ y = 9t - t^3 = t(9 - t^2)$; then $t^2 = \frac{x+4}{5}$ and also $\frac{y}{t} - 9 = -t^2$; so $\frac{x+4}{5} = 9 - \frac{y}{t}$ and hence $\frac{y}{t} = 9 - \frac{x+4}{5} = \frac{45-x-4}{5} = \frac{41-x}{5}$,

so that $t = \frac{5y}{41-x}$; then , substituting back into $x = 5t^2 - 4$, we have

$x = 5\left(\frac{5y}{41-x}\right)^2 - 4$, and hence $(x + 4)(41 - x)^2 = 125y^2$

Methods for converting from parametric to Cartesian form

(a) Make t the subject of one of the equations for x or y, and substitute for t in the other equation.

(b) Combine the equations for x & y in some way, so as to make t the subject (as in (i)).

(c) Make $f(t)$ the subject of both of the equations for x & y, and equate the two expressions (as in (ii), with $f(t) = t^2$), leaving perhaps a single t in the resulting equation.

(5) If $\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$, $\phi = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}$ and $w = \frac{u+v}{1+\frac{uv}{c^2}}$,

show that $\left(1 + \frac{uv}{c^2}\right)\gamma\phi = \frac{1}{\sqrt{1-\left(\frac{w}{c}\right)^2}}$

Solution

The required result is equivalent to

$$\left(1 + \frac{uv}{c^2}\right)^2\left(1 - \left(\frac{w}{c}\right)^2\right) = \left(1 - \left(\frac{u}{c}\right)^2\right)\left(1 - \left(\frac{v}{c}\right)^2\right)$$

or $\left(1 + \frac{uv}{c^2}\right)^2\left(1 - \left(\frac{w}{c}\right)^2\right) - \left(1 - \left(\frac{u}{c}\right)^2\right)\left(1 - \left(\frac{v}{c}\right)^2\right) = 0$

$$LHS = \left\{1 + \frac{2uv}{c^2} + \frac{(uv)^2}{c^4}\right\} - \frac{(u+v)^2}{c^2} - \{1 - \frac{u^2}{c^2} - \frac{v^2}{c^2} + \frac{(uv)^2}{c^4}\}$$

$= 0$, as required.

[This is a result from Special Relativity: if spaceship C is seen by spaceship B to be moving away from it at speed v, and spaceship B is seen by spaceship A to be moving away from it (in the same direction as previously) at speed u, then Newtonian Physics gives the speed of C relative to A as just $u + v$, but according to Special Relativity it is w.

$\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$ is the Lorentz factor associated with changes in measurements of time and length for an object moving at relative speed v.]

(6) (i) Find an expansion for $(a+b+c)^3$, and give a justification for the coefficients.

(ii) Extend this to $(a+b+c)^4$

Solution

(i) By an ordinary expansion:

$(a+b+c)^3 = ([a+b]+c)^3$

$= (a+b)^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3$

$= (a^3 + 3a^2b + 3ab^2 + b^3) + (3a^2c + 3b^2c + 6abc)$

$+(3ac^2 + 3bc^2) + c^3$

$= (a^3 + b^3 + c^3) + 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b)$

$+6abc$

Alternatively, this could have been deduced by noting that the terms fall into one of the 3 groups above.

Then there is only 1 way of creating an a^3 term from

$(a+b+c)(a+b+c)(a+b+c)$; namely by choosing the a from each of the 3 brackets.

There are 3 ways of creating an a^2b term: 3[number of ways of choosing the b]$\times$ 1[number of ways of choosing two as from the remaining 2 brackets].

Finally, there are 6 ways of creating an abc term: 3[number of ways of choosing the a]$\times$ 2[number of ways of choosing the b from the remaining 2 brackets]$\times$ 1[number of ways of choosing the c from the remaining bracket].

The final expression then follows by symmetry.

(ii) $(a+b+c)^4 = (a^4+b^4+c^4)$

$+4(a^3b+a^3c+b^3a+b^3c+c^3a+c^3b)$

$+6(a^2b^2+a^2c^2+b^2c^2)+12(a^2bc+b^2ac+c^2ab)$

For the a^2b^2 term etc, there are $\binom{4}{2}=6$ ways of choosing the brackets from $(a+b+c)(a+b+c)(a+b+c)(a+b+c)$

to give a^2, and then just 1 way of obtaining the b^2 term.

For the a^2bc term etc, there are $\binom{4}{2}=6$ ways of choosing the brackets for the a^2 again, multiplied by the 2 ways of choosing brackets for the b and c.

For further investigation: the 'trinomial' expansion of $(a+b+c)^n$ can be shown to be $\sum_{\substack{i,j,k \\ (i+j+k=n)}} \binom{n}{i,j,k} a^i b^j c^k$,

where $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$

(with a further extension to the 'multinomial' expansion of

$(a_1+a_2+\cdots+a_m)^n$)