STEP - Integration Exercises (sol'ns) (3 pages; 28/9/18)

(1) If $\int_{-a}^{a} f(x)\, dx = b$, find $\int_{-a}^{a} f(-x)\, dx$

Solution

b also, as $f(-x)$ is the reflection of $f(x)$ in the y-axis

(2) Explain the following 'paradox':

$\int \frac{1}{2\text{x}} dx = \frac{1}{2} \int \frac{1}{x} dx = \frac{1}{2} lnx + C$

but $\int \frac{1}{2\text{x}} dx = \frac{1}{2} ln(2x) + C$ (by the reverse Chain rule)

Solution

$ln(2x)$ can be written as $ln2 + lnx$, giving the first form of the answer, after renaming the constant

(3) Given that $\int \frac{1}{x} dx = \ln x$ for $x > 0$, show that $\int \frac{1}{x} dx = ln|x|$ for all $x \neq 0$

Solution

Method 1

If $\int \frac{1}{x} dx = \ln x$ for $x > 0$, then $\frac{d}{dx}(lnx) = \frac{1}{x}$ for $x > 0$

For the case where $x < 0$:

Let $y = -x$, so that $\frac{d}{dy}(lny) = \frac{1}{y}$, as $y > 0$

[To convert back to xs:]

Then, as $\frac{d}{dy}(lny) = \frac{d}{dx}(lny).\frac{dx}{dy}$,

it follows that $\frac{d}{dx}(lny).\frac{dx}{dy} = \frac{1}{(-x)}$

giving $\frac{d}{dx}(ln[-x])(-1) = \frac{1}{(-x)}$

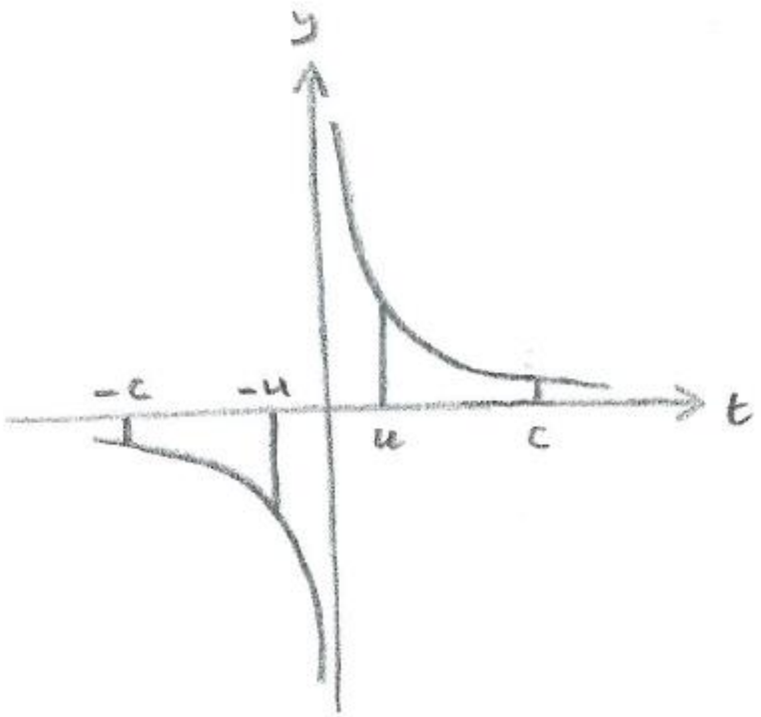
and so $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for $x < 0$ (*)

and therefore $\int \frac{1}{x} dx = \ln|x|$ for $x < 0$, as well as $x > 0$

[Note that the function $y = \ln|x|$ for $x < 0$ is the reflection in the y-axis of $y = \ln x$ $(for\ x > 0)$, and therefore has a negative gradient, which agrees with (*).]

Method 2

Referring to the diagram below, where $u = -x > 0$ & $c > 0$,



$\int_{-c}^{x} \frac{1}{t} dt = \int_{-c}^{-u} \frac{1}{t} dt$

$= -$ (positive) area between graph and t-axis on LHS

$= -$ (positive) area between graph and t-axis on RHS

$= -\int_u^c \frac{1}{t}dt = \int_c^u \frac{1}{t}dt = lnu - lnc$

As $\int \frac{1}{x}dx$ only differs from $\int_{-c}^{x} \frac{1}{t}dt$ by an arbitrary constant, it follows that, when $x < 0$, $\int \frac{1}{x}dx = ln\, u + C = \ln|-x| + C$, as

(4) Given that $f(x)$ has a maximum on the interval $0 \leq x \leq \frac{1}{2}$ at $x = x_0$, show that $\int_0^x f(t)dt \leq \frac{1}{2}f(x_0)$ whenever $0 \leq x \leq \frac{1}{2}$

Solution

Consider the area under the graph of $f(t)$, between 0 & x.

Assume for the moment that the graph lies above the t-axis.

The maximum height of the function is $f(x_0)$, and the area under the graph is no greater than the rectangle with base x and height $f(x_0)$.

As $x \leq \frac{1}{2}$, the rectangle has area $\leq \frac{1}{2}f(x_0)$.

As the integral would have a smaller value if part of the graph were to lie below the t-axis,

$\int_0^x f(t)dt \leq \frac{1}{2}f(x_0)$ whenever $0 \leq x \leq \frac{1}{2}$